Volumes: Slabs, Washers, Shells

Calculus II

Josh Engwer

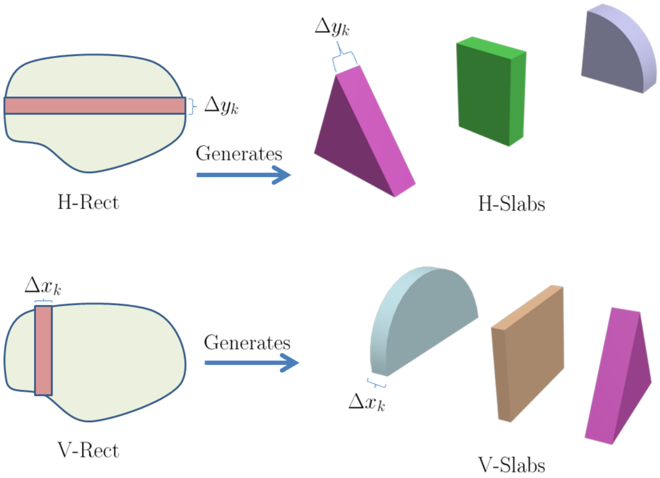
TTU

27 January 2014

VOLUMES PART I:

SOLIDS WITH KNOWN CROSS SECTIONS

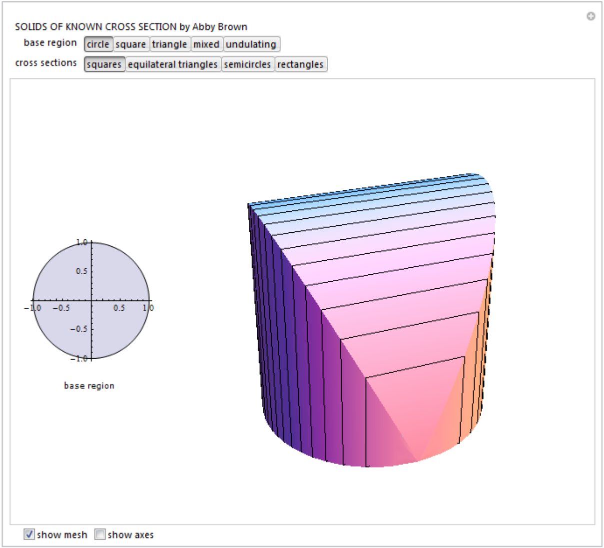
Solids with Known Cross Sections (Slabs)



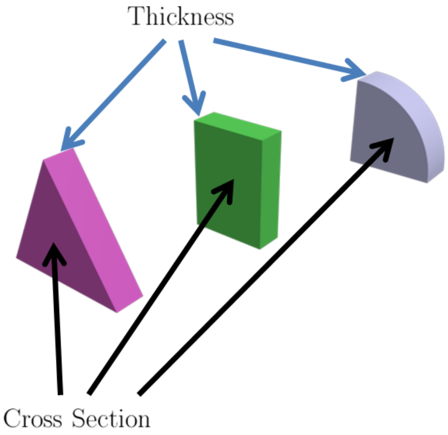
$$\left(\textbf{Thickness} \text{ of } k^{th} \text{ H-Slab}\right) = \left(\textbf{Width} \text{ of } k^{th} \text{ generating H-Rect}\right) = \Delta y_k$$

$$\left(\textbf{Thickness} \text{ of } k^{th} \text{ V-Slab}\right) = \left(\textbf{Width} \text{ of } k^{th} \text{ generating V-Rect}\right) = \Delta x_k$$

(DEMO) SOLIDS WITH KNOWN CROSS SECTION (Click below):



Volumes of Slabs



Proposition

Volume of Slab $= \Big($*Area of Cross Section*$\Big) \times \Big($*Thickness*$\Big)$

Volume of Solid with Known Cross Section (CS)

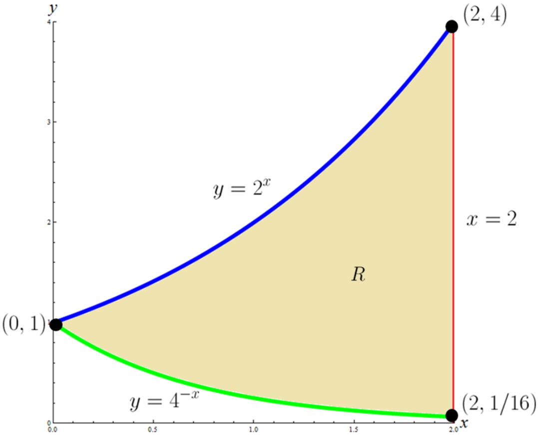
EXAMPLE: Let region R be bounded by curves $y = 2^x$, $y = 4^{-x}$, $x = 2$.

Let solid S have base R and equilateral triangular cross sections $\perp$ to x-axis.

Setup integral(s) to compute Volume(S).

$\perp \equiv$ "perpendicular"

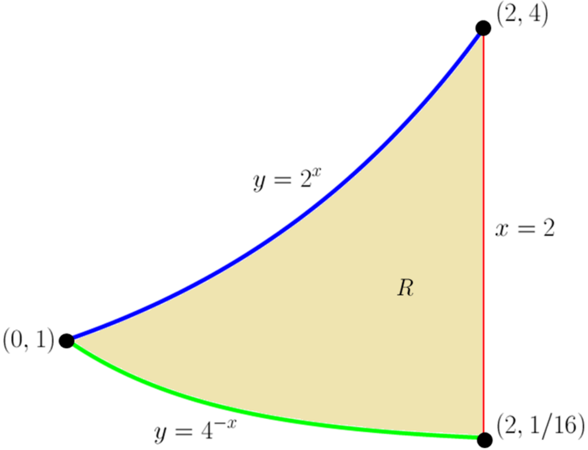
EXAMPLE: Let region R be bounded by curves $y = 2^x$, $y = 4^{-x}$, $x = 2$.



Sketch & characterize region R (label BP's & BC's in terms of x)
Notice region R is V-simple.

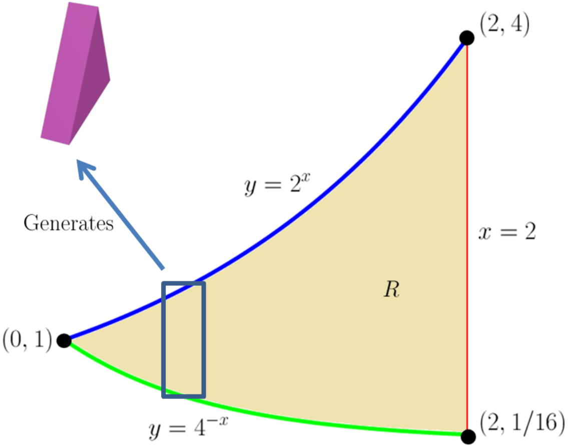
Volume of Solid with Known Cross Section (CS)

EXAMPLE: Let region R be bounded by curves $y = 2^x$, $y = 4^{-x}$, $x = 2$.



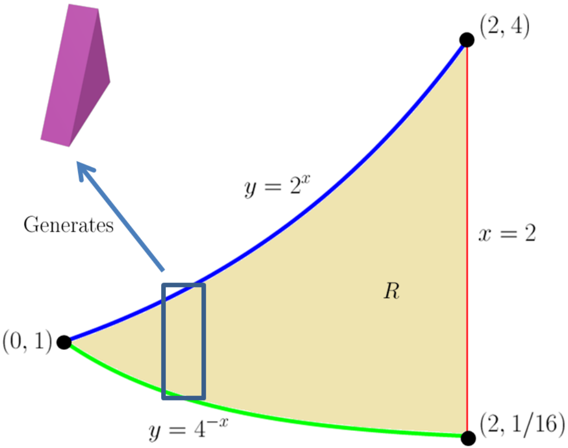
Sketch & characterize region R. (remove unnecessary clutter)
Notice region R is V-simple.

Volume of Solid with Known Cross Section (CS)



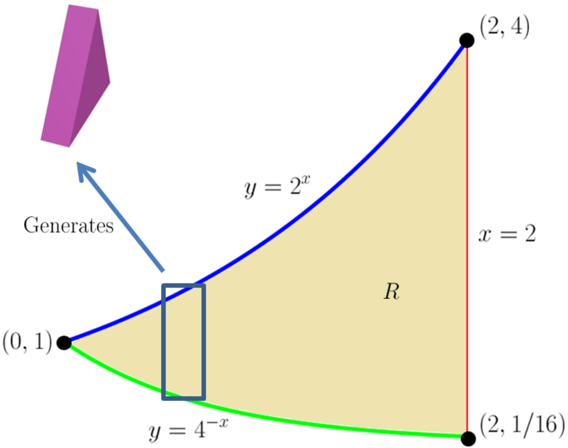
Key Element: V-Slab (CS = Equilateral Triangle) generated by V-Rect

Volume of Solid with Known Cross Section (CS)



$$k^{th} \text{ V-Slab on } R: \begin{array}{lcl} \text{Thickness} & = & \text{(Width of generating V-Rect)} \\ \text{Side Length} & = & \text{(Height of generating V-Rect)} \\ \hline \text{Volume} & = & \frac{\sqrt{3}}{4} \times \text{(Side Length)}^2 \times \text{(Thickness)} \end{array}$$

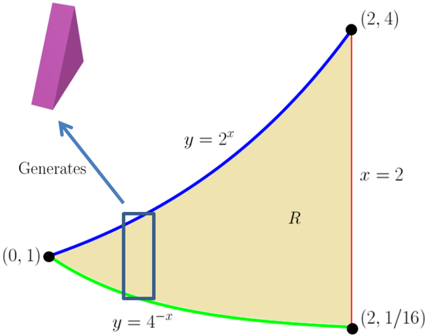
Volume of Solid with Known Cross Section (CS)



k^{th} V-Slab on R:

Thickness	$=$	Δx_k
Side Length	$=$	$2^{x_k^*} - 4^{-x_k^*}$
Volume	$=$	$\frac{\sqrt{3}}{4}\left[2^{x_k^*} - 4^{-x_k^*}\right]^2 \Delta x_k$

Volume of Solid with Known Cross Section (CS)



k^{th} V-Slab on R:

Thickness	$=$	Δx_k
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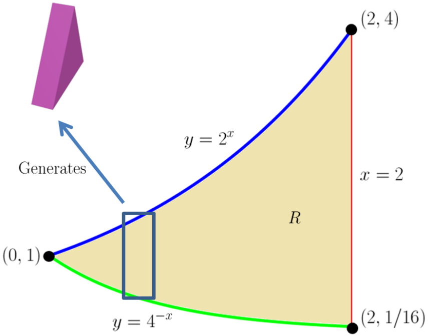
Volume	$=$	$\frac{\sqrt{3}}{4}\left[2^{x_k^*} - 4^{-x_k^*}\right]^2 \Delta x_k$

Riemann Sum: Volume(S) $\approx V_N^* = \sum_{k=1}^{N} \frac{\sqrt{3}}{4}\left[2^{x_k^*} - 4^{-x_k^*}\right]^2 \Delta x_k$

Volume of Solid with Known Cross Section (CS)



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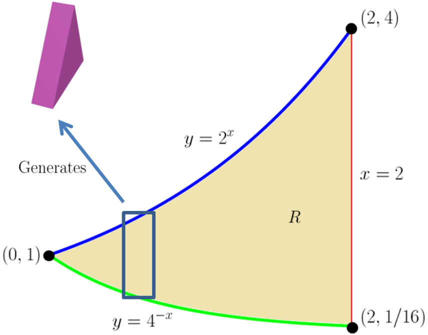
Integral: Volume(S) $= \lim_{N\to\infty} V_N^* = \int_{\text{smallest } x\text{-coord. in } R}^{\text{largest } x\text{-coord. in } R} \frac{\sqrt{3}}{4} \left(2^x - 4^{-x} \right)^2 \, dx$

Volume of Solid with Known Cross Section (CS)



Riemann Sum: Volume(S) $\approx V_N^* = \sum_{k=1}^{N} \frac{\sqrt{3}}{4} \left[2^{x_k^*} - 4^{-x_k^*} \right]^2 \Delta x_k$

Integral: Volume(S) $= \boxed{\int_0^2 \frac{\sqrt{3}}{4} \left(2^x - 4^{-x}\right)^2 \, dx} \approx 1.7954$

Volume of Solid with Known Cross Section (CS)

EXAMPLE: Let region R be bounded by curves $y = 2^x$, $y = 4^{-x}$, $x = 2$.

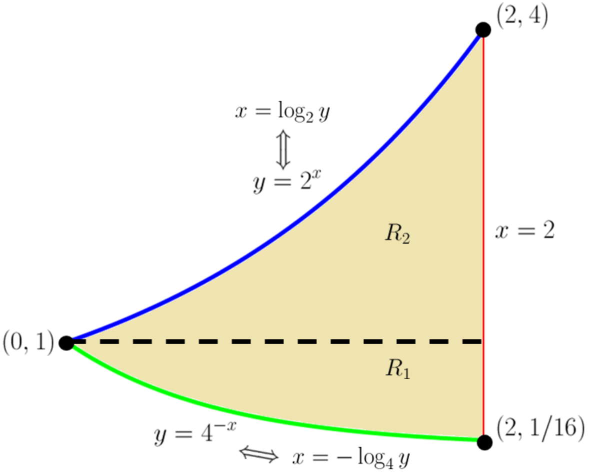
Let solid S have base R and equilateral triangular cross sections $\perp$ to y-axis.

Setup integral(s) to compute Volume(S).

$\perp \equiv$ "perpendicular"

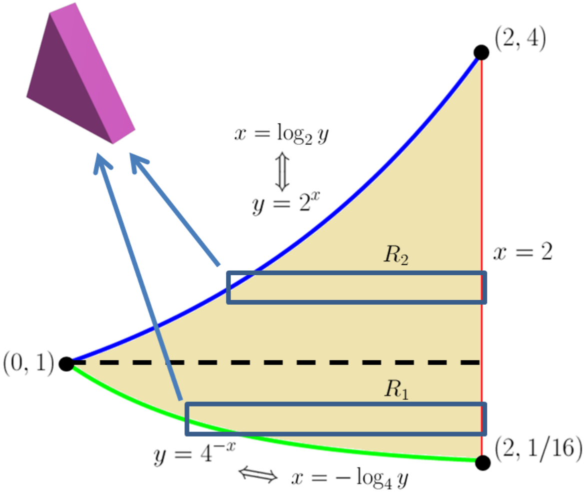
Volume of Solid with Known Cross Section (CS)

EXAMPLE: Let region R be bounded by curves $y = 2^x$, $y = 4^{-x}$, $x = 2$.



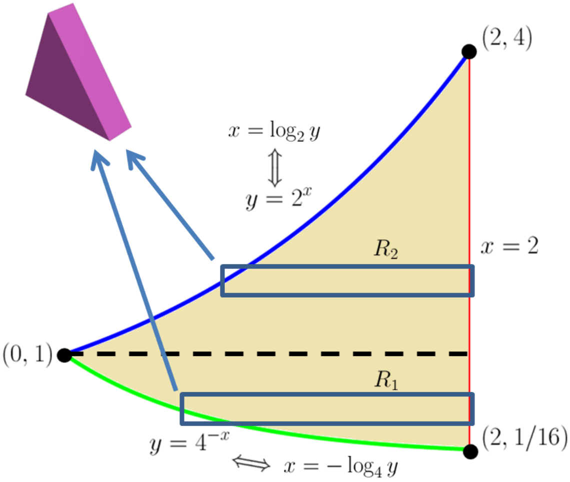
Sketch & characterize region R (label BP's & BC's in terms of y)
Notice region R is NOT H-simple, so subdivide into subregions R_1, R_2.

Volume of Solid with Known Cross Section (CS)



Key Element: H-Slab (CS = Equilateral Triangle) generated by H-Rect

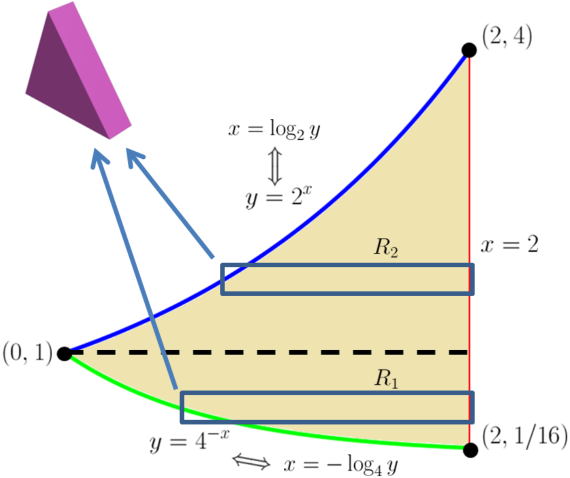
Volume of Solid with Known Cross Section (CS)



k^{th} H-Slab on R_1:

Thickness	$=$	(Width of generating H-Rect)
Side Length	$=$	(Length of generating H-Rect)
Volume	$=$	$\frac{\sqrt{3}}{4} \times (\text{Side Length})^2 \times (\text{Thickness})$

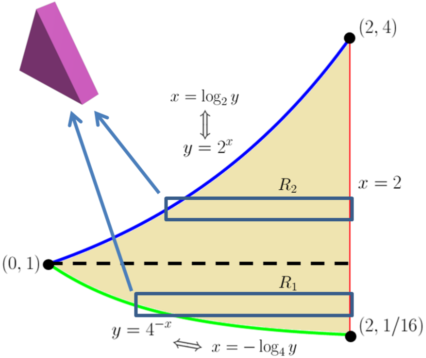
Volume of Solid with Known Cross Section (CS)



k^{th} H-Slab on R_1:

Thickness	$=$	Δy_k
Side Length	$=$	$2 - (-\log_4 y_k^*)$
Volume	$=$	$\frac{\sqrt{3}}{4}\left(2 + \log_4 y_k^*\right)^2 \Delta y_k$

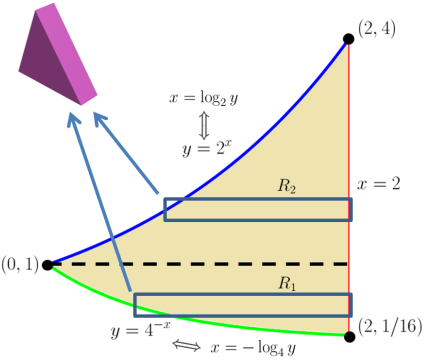
Volume of Solid with Known Cross Section (CS)



$$k^{th} \text{ H-Slab on } R_1: \quad \begin{array}{lcl} \text{Thickness} & = & \Delta y_k \\ \text{Side Length} & = & 2 - (-\log_4 y_k^*) \\ \hline \text{Volume} & = & \frac{\sqrt{3}}{4}\left(2 + \log_4 y_k^*\right)^2 \Delta y_k \end{array}$$

Riemann Sum: Volume(S_1) $\approx V_N^* = \sum_{k=1}^{N} \frac{\sqrt{3}}{4}\left(2 + \log_4 y_k^*\right)^2 \Delta y_k$

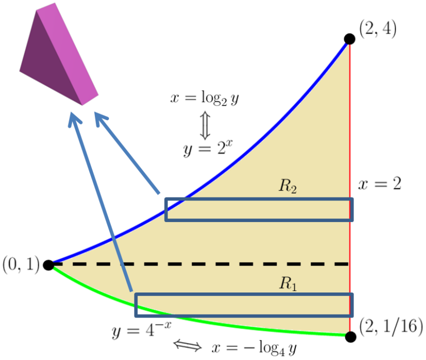
Volume of Solid with Known Cross Section (CS)



Riemann Sum: Volume(S_1) $\approx V_N^* = \sum_{k=1}^{N} \frac{\sqrt{3}}{4} (2 + \log_4 y_k^*)^2 \, \Delta y_k$

Integral: Volume(S_1) $= \lim_{N \to \infty} V_N^* = \int_{\text{smallest } y\text{-coord. in } R_1}^{\text{largest } y\text{-coord. in } R_1} \frac{\sqrt{3}}{4} (2 + \log_4 y)^2 \; dy$

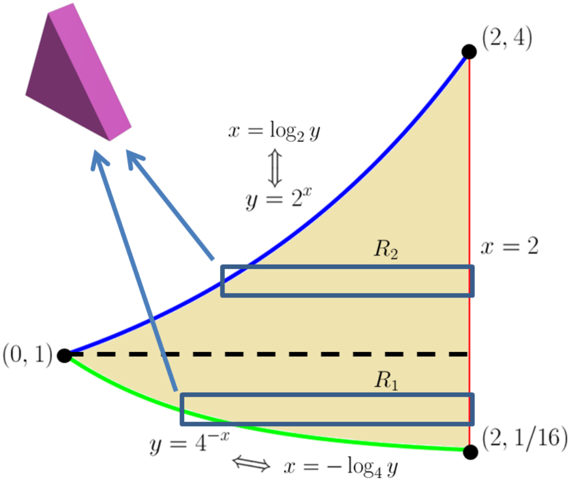
Volume of Solid with Known Cross Section (CS)



Riemann Sum: Volume(S_1) $\approx V_N^* = \sum_{k=1}^{N} \frac{\sqrt{3}}{4} \left(2 + \log_4 y_k^*\right)^2 \Delta y_k$

Integral: Volume(S_1) $= \int_{1/16}^{1} \frac{\sqrt{3}}{4} \left(2 + \log_4 y\right)^2 \, dy$

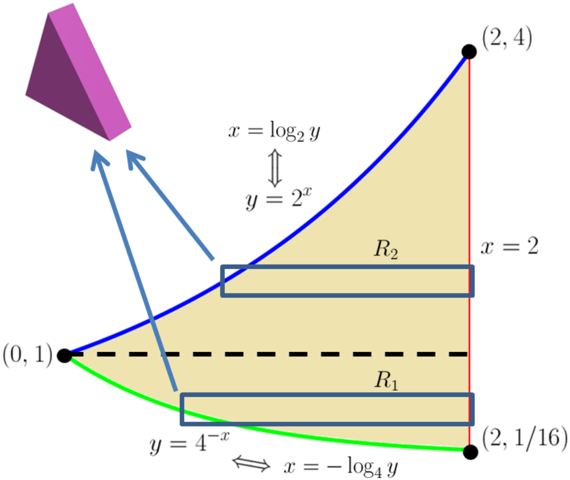
Volume of Solid with Known Cross Section (CS)



k^{th} H-Slab on R_2:

Thickness	=	(Width of generating H-Rect)
Side Length	=	(Length of generating H-Rect)
Volume	=	$\frac{\sqrt{3}}{4} \times (\text{Side Length})^2 \times (\text{Thickness})$

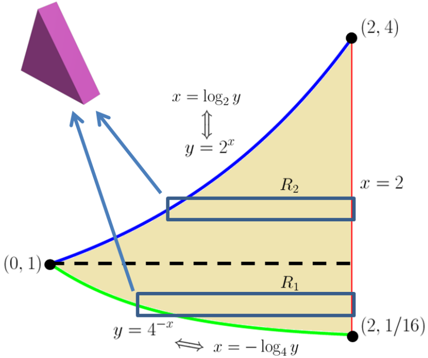
Volume of Solid with Known Cross Section (CS)



k^{th} H-Slab on R_2:

Thickness	$=$	Δy_k
Side Length	$=$	$2 - \log_2 y_k^*$
Volume	$=$	$\frac{\sqrt{3}}{4}\left(2 - \log_2 y_k^*\right)^2 \Delta y_k$

Volume of Solid with Known Cross Section (CS)

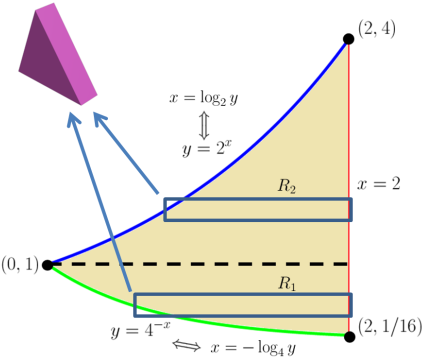


k^{th} H-Slab on R_2:

$$\begin{array}{lcl} \text{Thickness} & = & \Delta y_k \\ \text{Side Length} & = & 2 - \log_2 y_k^* \\ \hline \text{Volume} & = & \frac{\sqrt{3}}{4}\left(2 - \log_2 y_k^*\right)^2 \Delta y_k \end{array}$$

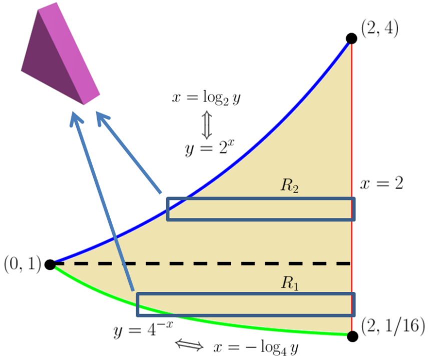
Riemann Sum: Volume(S_2) $\approx V_N^* = \displaystyle\sum_{k=1}^{N} \frac{\sqrt{3}}{4}\left(2 - \log_2 y_k^*\right)^2 \Delta y_k$

Volume of Solid with Known Cross Section (CS)



Riemann Sum: Volume(S_2) $\approx V_N^* = \sum_{k=1}^{N} \frac{\sqrt{3}}{4} \left(2 - \log_2 y_k^*\right)^2 \Delta y_k$

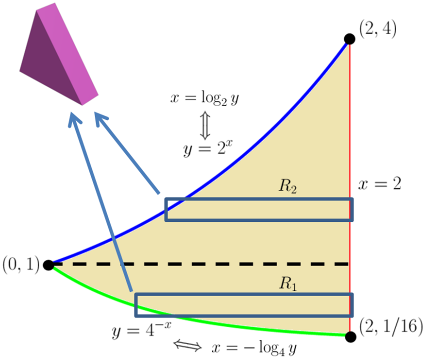
Integral: Volume(S_2) $= \lim_{N\to\infty} V_N^* = \int_{\text{smallest } y\text{-coord. in } R_2}^{\text{largest } y\text{-coord. in } R_2} \frac{\sqrt{3}}{4} \left(2 - \log_2 y\right)^2 \, dy$



Riemann Sum: Volume(S_2) $\approx V_N^* = \sum_{k=1}^{N} \frac{\sqrt{3}}{4} \left(2 - \log_2 y_k^*\right)^2 \Delta y_k$

Integral: Volume(S_2) $= \int_1^4 \frac{\sqrt{3}}{4} \left(2 - \log_2 y\right)^2 \, dy$

Volume of Solid with Known Cross Section (CS)



Region $R = R_1 \cup R_2 \implies$ Solid $S = S_1 \cup S_2$

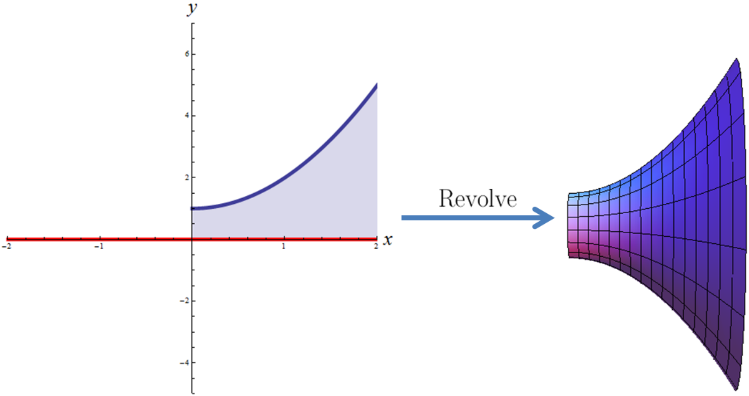
$\therefore$ Volume(S) = Volume(S_1) + Volume(S_2)

$$= \boxed{\int_{1/16}^{1} \frac{\sqrt{3}}{4} (2 + \log_4 y)^2 \; dy + \int_1^4 \frac{\sqrt{3}}{4} (2 - \log_2 y)^2 \; dy} \approx 2.0818$$

VOLUMES PART II:

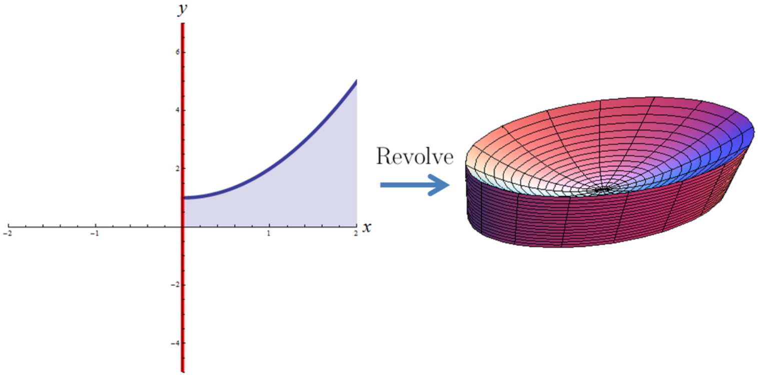
SOLIDS OF REVOLUTION

Solids of Revolution



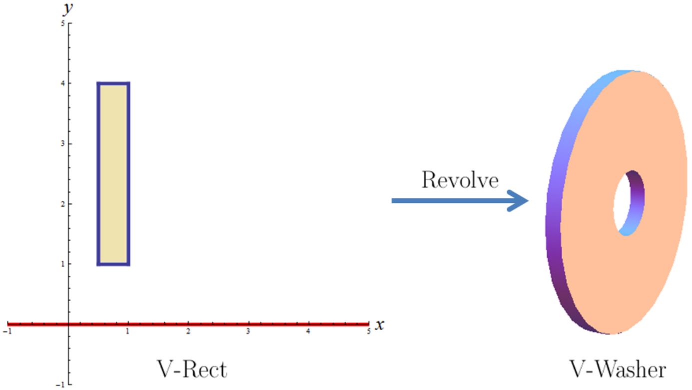
Axis of Revolution: x-axis

Solids of Revolution



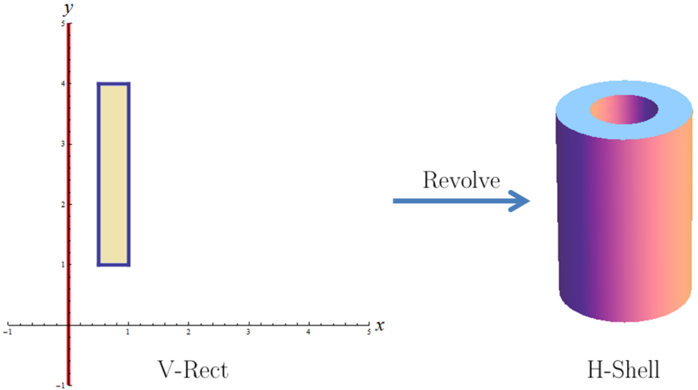
Axis of Revolution: y-axis

Solids of Revolution (V-Washers)



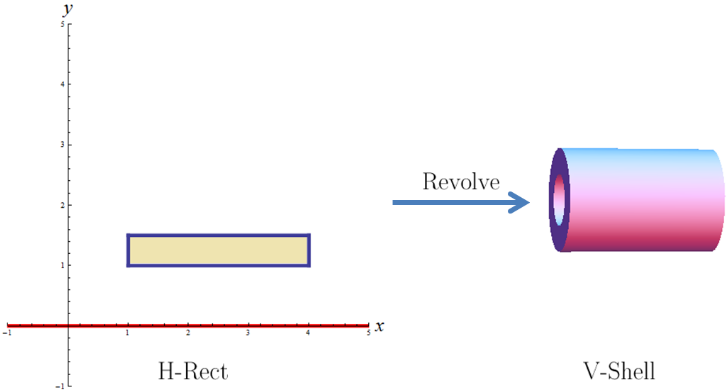
Axis of Revolution: x-axis

Solids of Revolution (H-Shells)



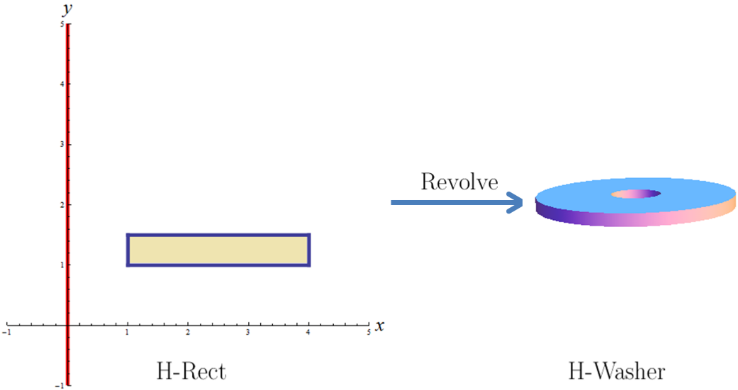
Axis of Revolution: y-axis

Solids of Revolution (V-Shells)



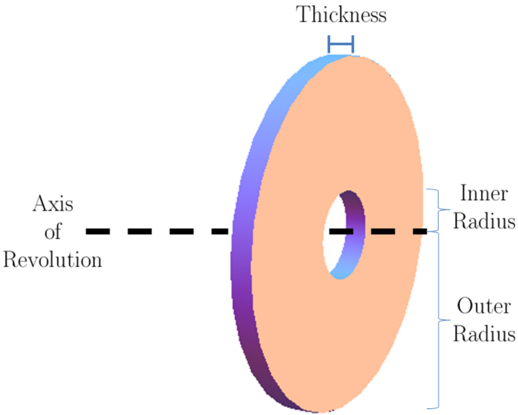
Axis of Revolution: x-axis

Solids of Revolution (H-Washers)



Axis of Revolution: y-axis

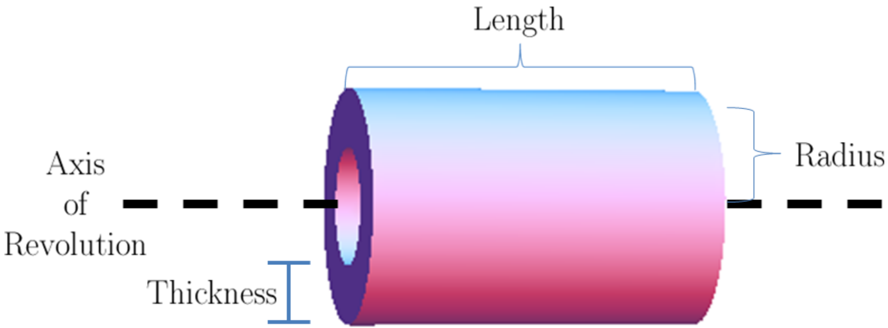
Volume of Washers



Proposition

$$\textit{Volume of Washer} = \pi \times \left[\left(\textit{Outer Radius}\right)^2 - \left(\textit{Inner Radius}\right)^2 \right] \times \left(\textit{Thickness}\right)$$

Volume of V-Shells

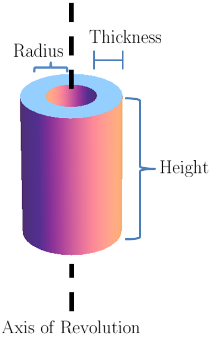


Proposition

Volume of k^{th} *V-Shell* $= 2\pi \times \Big(\textit{Radius}\Big) \times \Big(\textit{Length}\Big) \times \Big(\textit{Thickness}\Big)$

Radius involves the ***tag*** y_k^* *Thickness* $= \Delta y_k$

Volume of H-Shells



Proposition

Volume of k^{th} *H-Shell* $= 2\pi \times \Big(\textit{Radius}\Big) \times \Big(\textit{Height}\Big) \times \Big(\textit{Thickness}\Big)$

Radius involves the ***tag*** x_k^*

Thickness $= \Delta x_k$

Volume of Solid of Revolution (using Washers)

EXAMPLE: Let region R be bounded by curves $y = 2^x$, $y = 4^{-x}$, $x = 2$.

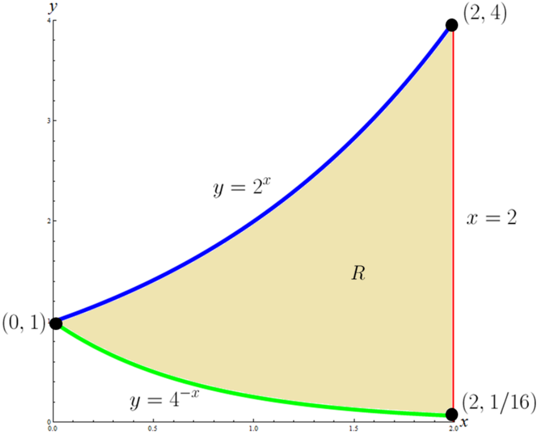
Let S be the solid formed by revolving region R about x-axis using washers.

Setup integral(s) to compute Volume(S).

$\perp \equiv$ "perpendicular"

Volume of Solid of Revolution (using Washers)

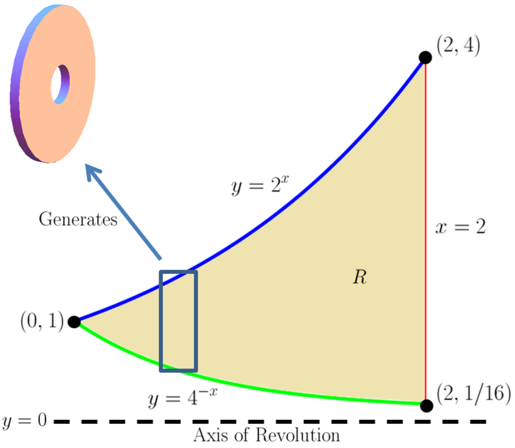
EXAMPLE: Let region R be bounded by curves $y = 2^x$, $y = 4^{-x}$, $x = 2$.



Sketch & characterize region R (label BP's & BC's in terms of x)
Notice region R is V-simple.

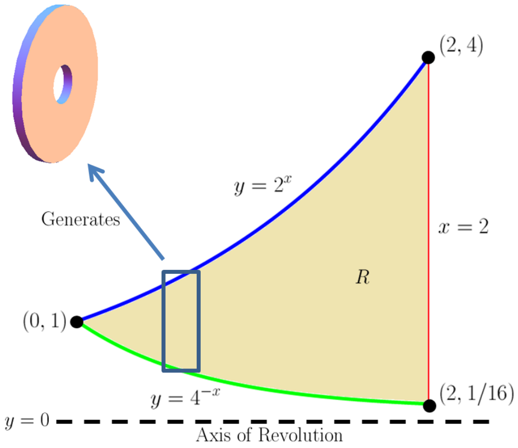
Volume of Solid of Revolution (using Washers)

EXAMPLE: Let region R be bounded by curves $y = 2^x$, $y = 4^{-x}$, $x = 2$.



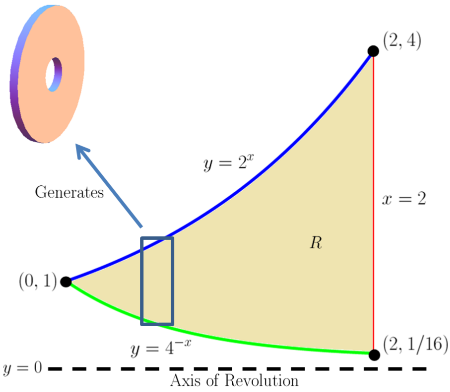
Sketch & characterize region R. (remove unnecessary clutter)
Sketch & label the **axis of revolution** (dashed line).

Volume of Solid of Revolution (using Washers)



Key Element: V-Washer (generated by V-Rect)

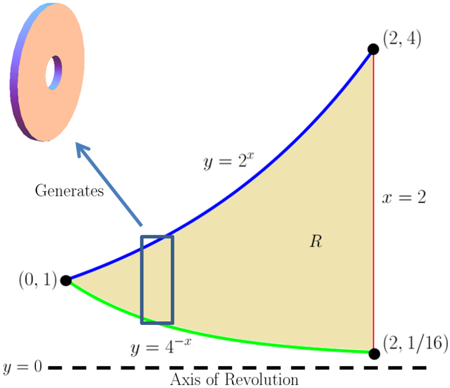
Volume of Solid of Revolution (using Washers)



k^{th} V-Washer on R:

Thickness	=	(Width of generating V-Rect)
Outer Radius	=	(Distance from farther BC to Axis of Revolution)
Inner Radius	=	(Distance from closer BC to Axis of Revolution)
Volume	=	$\pi \times \left[(\text{Outer Radius})^2 - (\text{Inner Radius})^2\right] \times (\text{Thickness})$

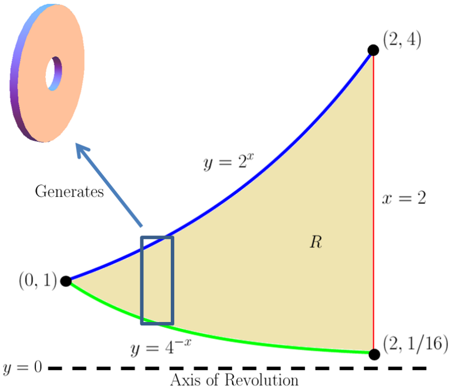
Volume of Solid of Revolution (using Washers)



k^{th} V-Washer on R:

Thickness	$=$	Δx_k
Outer Radius	$=$	$2^{x_k^*} - 0$
Inner Radius	$=$	$4^{-x_k^*} - 0$
Volume	$=$	$\pi \left[\left(2^{x_k^*}\right)^2 - \left(4^{-x_k^*}\right)^2 \right] \Delta x_k$

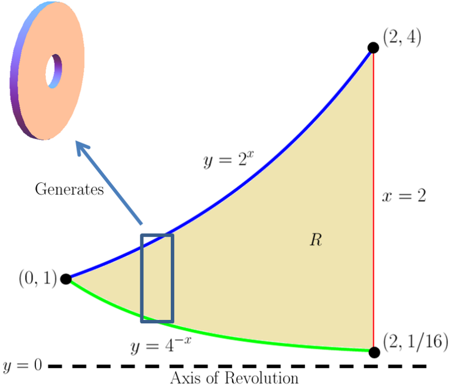
Volume of Solid of Revolution (using Washers)



k^{th} V-Washer on R:

Thickness	$=$	Δx_k
Outer Radius	$=$	$2^{x_k^*} - 0$
Inner Radius	$=$	$4^{-x_k^*} - 0$
Volume	$=$	$\pi \left[2^{2x_k^*} - 4^{-2x_k^*} \right] \Delta x_k$

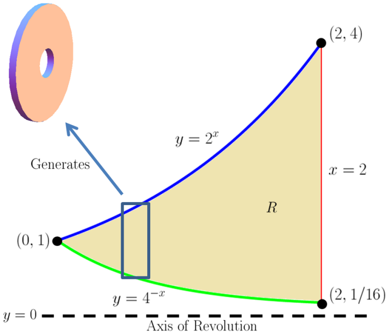
Volume of Solid of Revolution (using Washers)



k^{th} V-Washer on R:

Thickness	$=$	Δx_k
Outer Radius	$=$	$2^{x_k^*} - 0$
Inner Radius	$=$	$4^{-x_k^*} - 0$
Volume	$=$	$\pi\left[4^{x_k^*} - 16^{-x_k^*}\right]\Delta x_k$

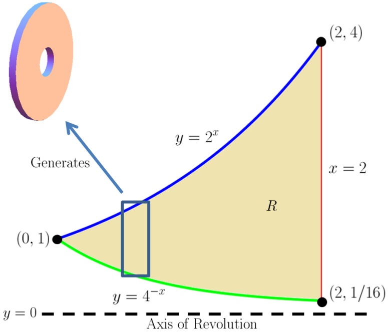
Volume of Solid of Revolution (using Washers)



$$k^{th} \text{ V-Washer on } R: \quad \begin{array}{lcl} \text{Thickness} & = & \Delta x_k \\ \text{Outer Radius} & = & 2^{x_k^*} - 0 \\ \text{Inner Radius} & = & 4^{-x_k^*} - 0 \\ \hline \text{Volume} & = & \pi \left[4^{x_k^*} - 16^{-x_k^*} \right] \Delta x_k \end{array}$$

Riemann Sum: Volume(S) $\approx V_N^* = \sum_{k=1}^{N} \pi \left[4^{x_k^*} - 16^{-x_k^*} \right] \Delta x_k$

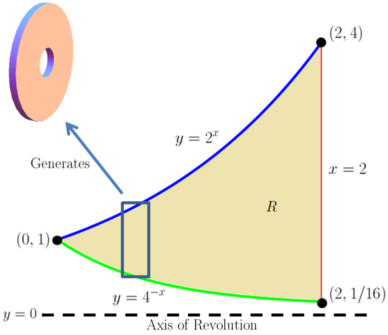
Volume of Solid of Revolution (using Washers)



Riemann Sum: Volume(S) $\approx V_N^* = \sum_{k=1}^{N} \pi \left[4^{x_k^*} - 16^{-x_k^*} \right] \Delta x_k$

Integral: Volume(S) $= \lim_{N \to \infty} V_N^* = \int_{\text{smallest } x\text{-coord. in } R}^{\text{largest } x\text{-coord. in } R} \pi \left(4^x - 16^{-x} \right) \, dx$

Volume of Solid of Revolution (using Washers)



Riemann Sum: Volume(S) $\approx V_N^* = \sum_{k=1}^{N} \pi \left[4^{x_k^*} - 16^{-x_k^*} \right] \Delta x_k$

Integral: Volume(S) $= \boxed{\int_0^2 \pi \left(4^x - 16^{-x} \right) \, dx} \approx 32.864$

Volume of Solid of Revolution (using Shells)

EXAMPLE: Let region R be bounded by curves $y = 2^x$, $y = 4^{-x}$, $x = 2$.

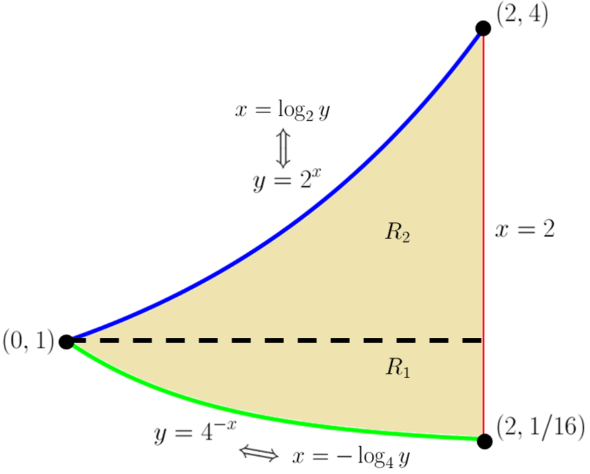
Let S be the solid formed by revolving region R about line $y = 5$ using shells.

Setup integral(s) to compute Volume(S).

$\perp \equiv$ "perpendicular"

Volume of Solid of Revolution (using Shells)

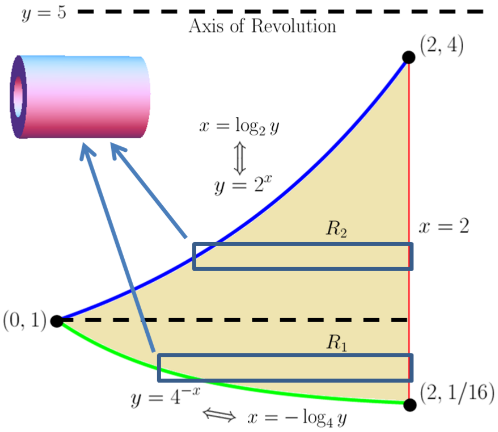
EXAMPLE: Let region R be bounded by curves $y = 2^x$, $y = 4^{-x}$, $x = 2$.



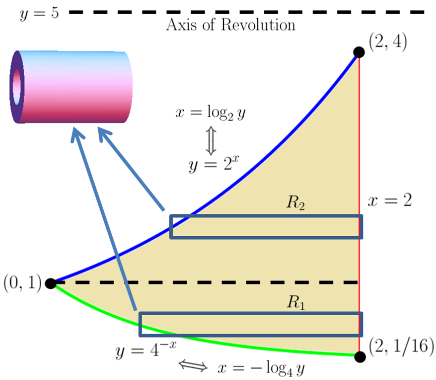
Sketch & characterize region R (label BP's & BC's in terms of y)
Notice region R is NOT H-simple, so subdivide into subregions R_1, R_2.

Volume of Solid of Revolution (using Shells)

EXAMPLE: Let region R be bounded by curves $y = 2^x$, $y = 4^{-x}$, $x = 2$.



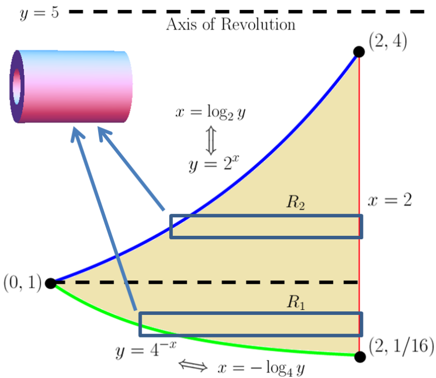
Key Element: V-Shell (generated by H-Rect)
Sketch & label the **axis of revolution** $y = 5$.



k^{th} V-Shell on R_1:

Thickness	$=$	(Width of generating H-Rect)
Radius	$=$	(Positive Distance from H-Rect to Axis of Revolution)
Length	$=$	(Length of generating H-Rect)
Volume	$=$	$2\pi \times$ (Radius) $\times$ (Length) $\times$ (Thickness)

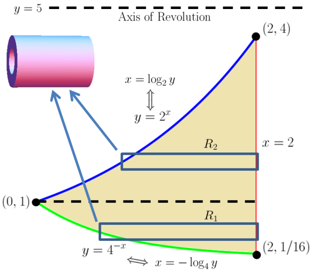
Volume of Solid of Revolution (using Shells)



k^{th} V-Shell on R_1:

$$\begin{aligned} \text{Thickness} &= \Delta y_k \\ \text{Radius} &= 5 - y_k^* \\ \text{Length} &= 2 - (-\log_4 y_k^*) \\ \hline \text{Volume} &= 2\pi\,(5 - y_k^*)\,(2 + \log_4 y_k^*)\,\Delta y_k \end{aligned}$$

Volume of Solid of Revolution (using Shells)



k^{th} V-Shell on R_1:

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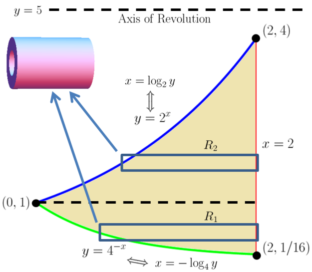
Riemann Sum: Volume$(S_1) \approx V_N^* = \sum_{k=1}^{N} 2\pi\,(5 - y_k^*)\,(2 + \log_4 y_k^*)\,\Delta y_k$

Volume of Solid of Revolution (using Shells)



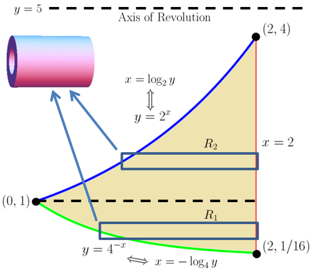
k^{th} V-Shell on R_1:

Thickness	$=$	Δy_k
Radius	$=$	$5 - y_k^*$
Length	$=$	$2 - (-\log_4 y_k^*)$
Volume	$=$	$2\pi\,(5 - y_k^*)\,(2 + \log_4 y_k^*)\,\Delta y_k$

Riemann Sum: Volume(S_1) $\approx V_N^* = \sum_{k=1}^{N} 2\pi\,(5 - y_k^*)\,(2 + \log_4 y_k^*)\,\Delta y_k$

Integral: Volume(S_1) $= \lim_{N\to\infty} V_N^* = \int_{\text{smallest } y\text{-coord. in } R_1}^{\text{largest } y\text{-coord. in } R_1} 2\pi\,(5 - y)\,(2 + \log_4 y)\;dy$

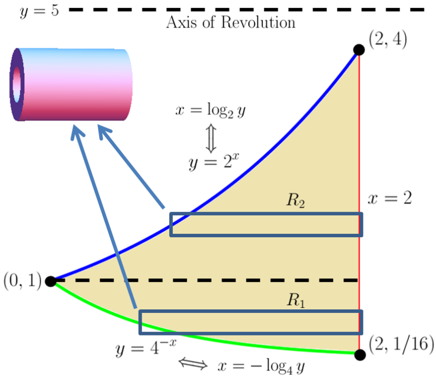
Volume of Solid of Revolution (using Shells)



Riemann Sum: Volume(S_1) $\approx V_N^* = \sum_{k=1}^{N} 2\pi \left(5 - y_k^*\right)\left(2 + \log_4 y_k^*\right) \Delta y_k$

Integral: Volume(S_1) $= \int_{1/16}^{1} 2\pi \left(5 - y\right)\left(2 + \log_4 y\right)\ dy$

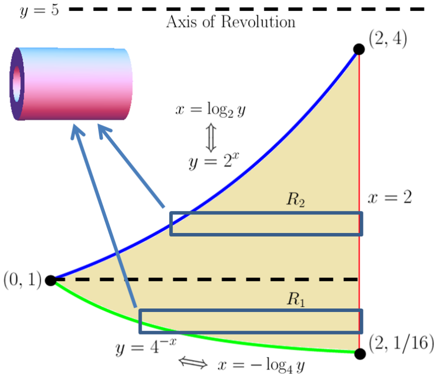
Volume of Solid of Revolution (using Shells)



k^{th} V-Shell on R_2:

Thickness	=	(Width of generating H-Rect)
Radius	=	(Positive Distance from H-Rect to Axis of Revolution)
Length	=	(Length of generating H-Rect)
Volume	=	$2\pi \times$ (Radius) $\times$ (Length) $\times$ (Thickness)

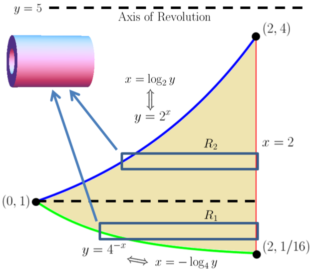
Volume of Solid of Revolution (using Shells)



k^{th} V-Shell on R_2:

$$\begin{array}{lcl} \text{Thickness} & = & \Delta y_k \\ \text{Radius} & = & 5 - y_k^* \\ \text{Length} & = & 2 - \log_2 y_k^* \\ \hline \text{Volume} & = & 2\pi \left(5 - y_k^*\right)\left(2 - \log_2 y_k^*\right)\Delta y_k \end{array}$$

Volume of Solid of Revolution (using Shells)

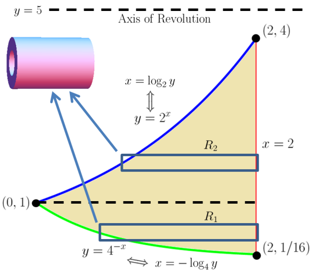


k^{th} V-Shell on R_2:

$$\begin{array}{lcl} \text{Thickness} & = & \Delta y_k \\ \text{Radius} & = & 5 - y_k^* \\ \text{Length} & = & 2 - \log_2 y_k^* \\ \hline \text{Volume} & = & 2\pi\left(5 - y_k^*\right)\left(2 - \log_2 y_k^*\right)\Delta y_k \end{array}$$

Riemann Sum: $\text{Volume}(S_2) \approx V_N^* = \sum_{k=1}^{N} 2\pi\left(5 - y_k^*\right)\left(2 - \log_2 y_k^*\right)\Delta y_k$

Volume of Solid of Revolution (using Shells)



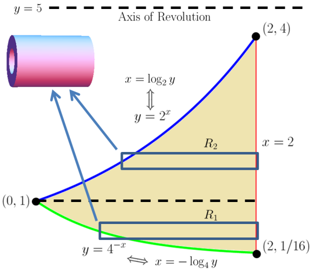
k^{th} V-Shell on R_2:

$$\begin{array}{lcl} \text{Thickness} & = & \Delta y_k \\ \text{Radius} & = & 5 - y_k^* \\ \text{Length} & = & 2 - \log_2 y_k^* \\ \hline \text{Volume} & = & 2\pi \left(5 - y_k^*\right)\left(2 - \log_2 y_k^*\right)\Delta y_k \end{array}$$

Riemann Sum: Volume(S_2) $\approx V_N^* = \displaystyle\sum_{k=1}^{N} 2\pi \left(5 - y_k^*\right)\left(2 - \log_2 y_k^*\right)\Delta y_k$

Integral: Volume(S_2) $= \displaystyle\lim_{N\to\infty} V_N^* = \int_{\text{smallest } y\text{-coord. in } R_2}^{\text{largest } y\text{-coord. in } R_2} 2\pi \left(5 - y\right)\left(2 - \log_2 y\right)\, dy$

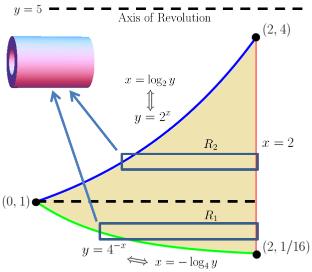
Volume of Solid of Revolution (using Shells)



Riemann Sum: Volume(S_2) $\approx V_N^* = \sum_{k=1}^{N} 2\pi \left(5 - y_k^*\right)\left(2 - \log_2 y_k^*\right)\Delta y_k$

Integral: Volume(S_2) $= \int_1^4 2\pi \left(5 - y\right)\left(2 - \log_2 y\right)\, dy$

Volume of Solid of Revolution (using Shells)



Region $R = R_1 \cup R_2 \implies$ Solid $S = S_1 \cup S_2$

$\therefore$ Volume(S) = Volume(S_1) + Volume(S_2)

$$= \boxed{\int_{1/16}^{1} 2\pi\,(5-y)\,(2+\log_4 y)\ dy + \int_{1}^{4} 2\pi\,(5-y)\,(2-\log_2 y)\ dy} \approx 81.8613$$

Fin.