Polar Curves: Graphs & Areas

Calculus II

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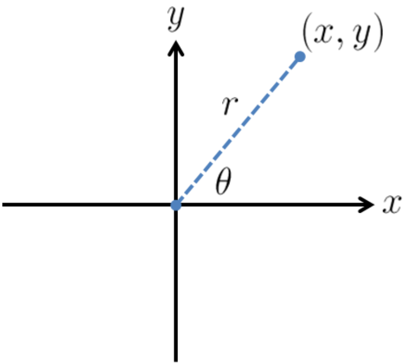
TTU

3 February 2014

POLAR CURVES PART I:
GRAPHS OF POLAR CURVES

Convert: Rectangular Coord's $\leftrightarrow$ Polar Coord's



$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \iff \begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = y/x \end{cases}$$

REMARK: r can be **negative**.

Converting Polar Coord. → Rectangular Coord.

WORKED EXAMPLE: Convert polar coordinate $(3, \pi/4)$ to rectangular form.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \implies \begin{cases} x = 3\cos(\pi/4) = (3)\left(1/\sqrt{2}\right) \\ y = 3\sin(\pi/4) = (3)\left(1/\sqrt{2}\right) \end{cases} \implies \boxed{\left(\frac{3}{\sqrt{2}}, \frac{3}{\sqrt{2}}\right)}$$

WORKED EXAMPLE: Convert polar coord. $(-3, \pi/4)$ to rectangular form.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \implies \begin{cases} x = -3\cos(\pi/4) = (-3)\left(1/\sqrt{2}\right) \\ y = -3\sin(\pi/4) = (-3)\left(1/\sqrt{2}\right) \end{cases} \implies \boxed{\left(-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right)}$$

WORKED EXAMPLE: Convert polar coord. $(3, -\pi/4)$ to rectangular form.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \implies \begin{cases} x = 3\cos(-\pi/4) = (3)\left(1/\sqrt{2}\right) \\ y = 3\sin(-\pi/4) = (3)\left(-1/\sqrt{2}\right) \end{cases} \implies \boxed{\left(\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right)}$$

WORKED EXAMPLE: Convert polar coord. $(3, 5\pi/4)$ to rectangular form.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases} \implies \begin{cases} x = 3\cos(5\pi/4) = (3)\left(-1/\sqrt{2}\right) \\ y = 3\sin(5\pi/4) = (3)\left(-1/\sqrt{2}\right) \end{cases} \implies \boxed{\left(-\frac{3}{\sqrt{2}}, -\frac{3}{\sqrt{2}}\right)}$$

Special Polar Curves

$\left(a \neq 0, b \neq 0, k \neq 0, n \in \mathbb{Z}_+\right)$ $\mathbb{Z}_+ \equiv$ The set of all **positive integers**.

POLAR CURVE	PROTOTYPE	REMARK(S)
Rays thru Pole	$\theta = k$	Always Graph!
Horizontal Lines (Off-Pole)	$r = a \csc\theta$	Always convert!
Vertical Lines (Off-Pole)	$r = a \sec\theta$	Always convert!
Circles Centered at Pole	$r = k$	Always Graph!
Circles Containing Pole	$r = a\cos\theta, r = a\sin\theta$	Always Graph!
Cardioids	$r = a \pm a\cos\theta, r = a \pm a\sin\theta$	Always Graph!
Limaçons	$r = b \pm a\cos\theta, r = b \pm a\sin\theta$	Always Graph!
Roses	$r = a\cos(n\theta), r = a\sin(n\theta)$	Always Graph!
Lemniscates	$r^2 = a^2\cos(2\theta), r^2 = a^2\sin(2\theta)$	Always Graph!
Arithmetic Spirals	$r = k\theta$	Always Graph!

The **pole** is the origin.

$$r = a\csc\theta \iff r = \frac{a}{\sin\theta} \iff r\sin\theta = a \iff y = a$$

$$r = a\sec\theta \iff r = \frac{a}{\cos\theta} \iff r\cos\theta = a \iff x = a$$

Special Polar Curves NOT to be Considered

The following special polar curves are too subtle or complicated to graph and use with double integrals, and so will not be considered here:
$\left(a \neq 0, b \neq 0, k \neq 0\right)$

- Logarithmic Spiral: $r = ae^{b\theta}$, $r = a^{k\theta}$
- Strophoid: $r = a\cos(2\theta)\sec\theta$
- Bifolium: $r = a\sin\theta\cos^2\theta$
- Folium of Descartes: $r = \dfrac{3a\sin\theta\cos\theta}{\sin^3\theta + \cos^3\theta}$
- Ovals of Cassini: $r^4 + b^4 - 2b^2r^2\cos(2\theta) = k^4$

Graphing Polar Curves (Procedure)

1. Graph $r = f(\theta)$ on the usual xy-plane where $x = \theta$ & $y = r$ (**Rectangular Plot**)

   Use special angles for θ: $\left\{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi\right\}$

   If $f(\theta)$ has a trig fcn, set its argument to these angles & solve for θ:

   e.g. $f(\theta) = 7\sin(2\theta)$

   $\implies 2\theta \in \left\{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, 2\pi\right\}$

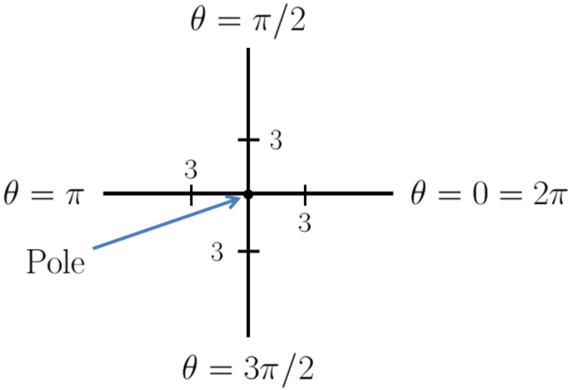
   $\implies \theta \in \left\{0, \frac{\pi}{8}, \frac{\pi}{4}, \frac{3\pi}{8}, \frac{\pi}{2}, \frac{5\pi}{8}, \frac{3\pi}{4}, \frac{7\pi}{8}, \pi\right\}$

2. Use the rectangular plot of $r = f(\theta)$ to trace the polar graph of $r = f(\theta)$ (**Polar Plot**)

   Stop when it's clear that continuing would re-trace the polar plot.

IMPORTANT: Except for equations of lines, "connect the dots" using **smooth curves**, not line segments!

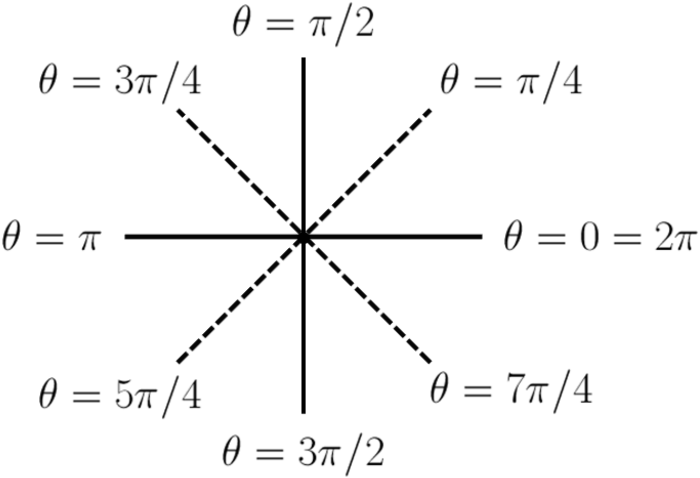
Polar Plots



REMARKS:

Even though r can be negative, only label key **positive** r-values on each ray.
The **pole** is the origin, but it has no unique polar representation: $(0, \theta)$
Polar coordinates are NOT unique: $\left(2, \frac{7\pi}{4}\right) = \left(2, -\frac{\pi}{4}\right) = \left(-2, \frac{3\pi}{4}\right) = \left(-2, -\frac{5\pi}{4}\right)$

Polar Plots



If necessary, include more rays in the polar plot.

Polar Plots (Example)

WORKED EXAMPLE: Plot the polar curve $r = -2\sin\theta$.

Polar Plots (Example)

WORKED EXAMPLE: Plot the polar curve $r = -2\sin\theta$.

STEP 1: Identify the **key θ-values**.

The argument of the trig fcn should use "easy angles":

$$\implies \theta \in \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\right\}$$

These are key values to label on the **horizontal axis** of **rectangular plot**.

Polar Plots (Example)

WORKED EXAMPLE: Plot the polar curve $r = -2\sin\theta$.

STEP 1: Identify the **key θ-values.**

The argument of the trig fcn should use "easy angles":

$$\implies \theta \in \left\{0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\right\}$$

These are key values to label on the **horizontal axis** of **rectangular plot**.

STEP 2: Identify the **key r-values.**

Find the **range** of the curve: $\text{Rng}\Big[-2\sin\theta\Big] = [-2, 2]$

The key values are the curve's max value, min value, & mid value:

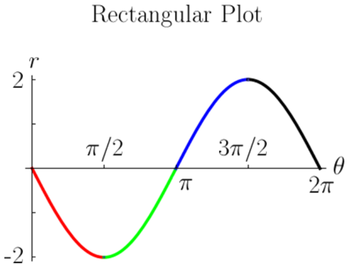
$$\text{Rng}\Big[-2\sin\theta\Big] = [-2, 2] \implies \begin{cases} \text{Max Value = 2} \\ \text{Min Value = -2} \end{cases}$$

$$\implies \text{Mid Value} := \frac{(\text{Max Value}) + (\text{Min Value})}{2} = \frac{2 + (-2)}{2} = 0$$

These are key values to label on the **vertical axis** of **rectangular plot**.

Polar Plots (Example)

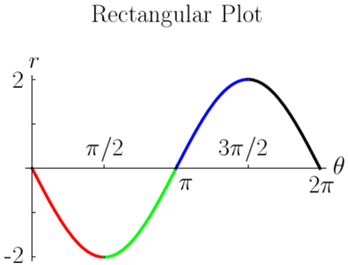
WORKED EXAMPLE: Plot the polar curve $r = -2\sin\theta$.

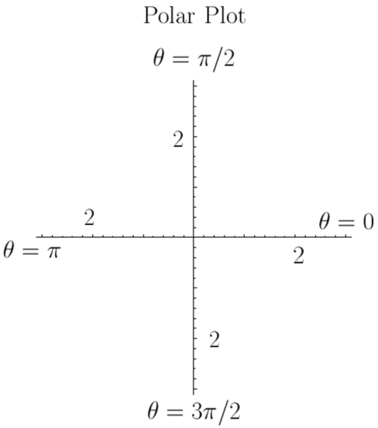


STEP 3: Trace the **rectangular plot** of $r = -2\sin\theta$

Polar Plots (Example)

WORKED EXAMPLE: Plot the polar curve $r = -2\sin\theta$.

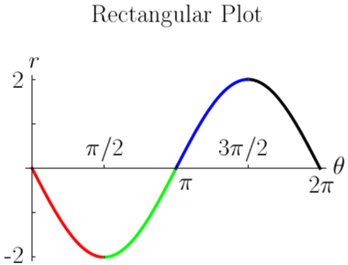


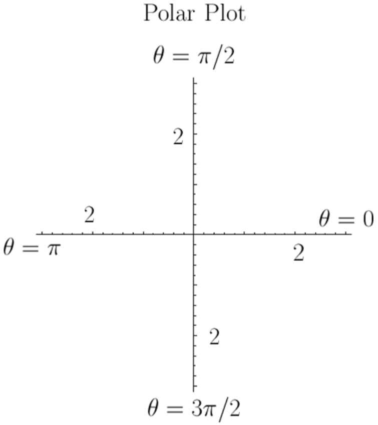


STEP 4: Trace the **polar plot**.

Polar Plots (Example)

WORKED EXAMPLE: Plot the polar curve $r = -2\sin\theta$.

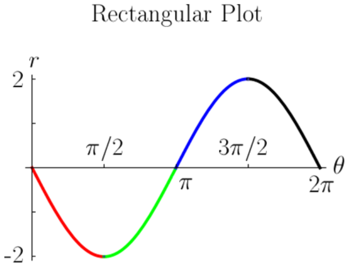


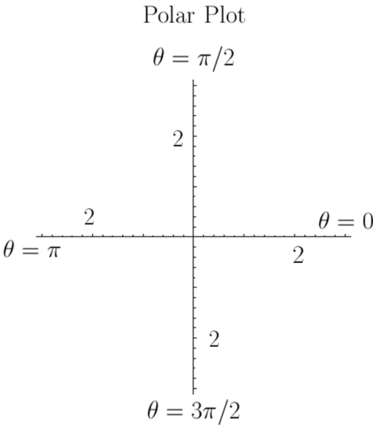


Setup the axes for the polar plot.

Polar Plots (Example)

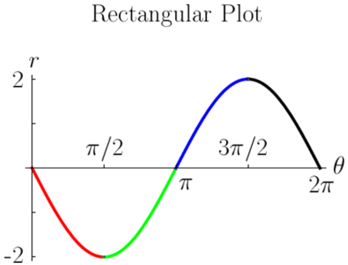
WORKED EXAMPLE: Plot the polar curve $r = -2\sin\theta$.

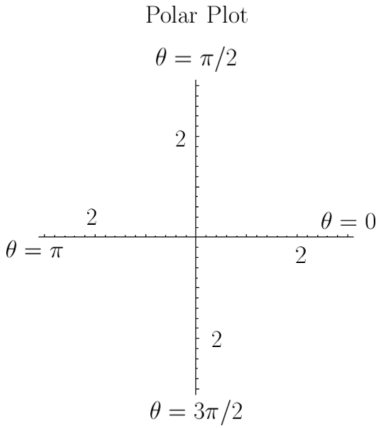




Consider the red portion of the rectangular plot (i.e. $\theta \in [0, \pi/2]$).

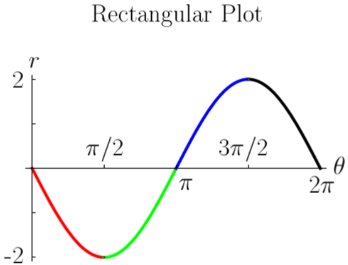
WORKED EXAMPLE: Plot the polar curve $r = -2\sin\theta$.

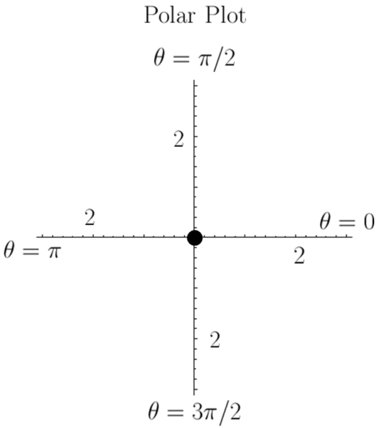




Starting Point: $\theta = 0 \implies r = 0$.

WORKED EXAMPLE: Plot the polar curve $r = -2\sin\theta$.

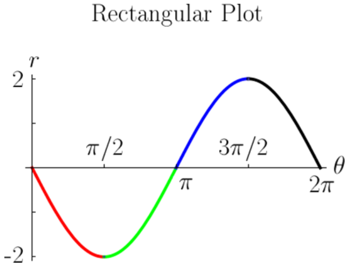


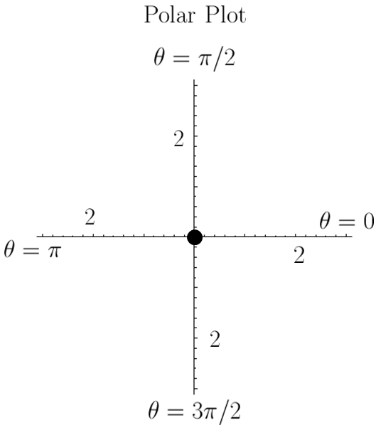


$$\theta = 0, r = 0 \implies$$ **pole.**

Polar Plots (Example)

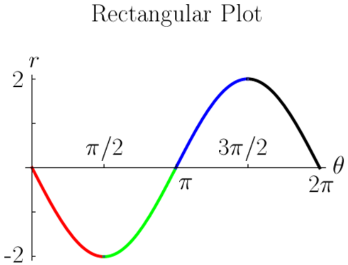
WORKED EXAMPLE: Plot the polar curve $r = -2\sin\theta$.

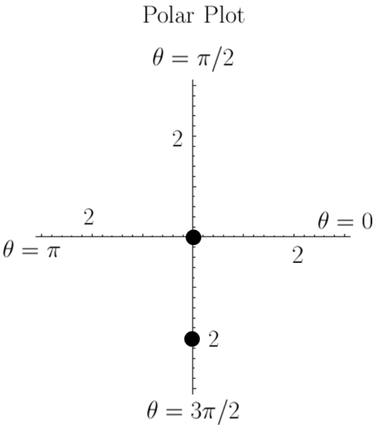




Ending Point: $\theta = \pi/2 \implies r = -2$.

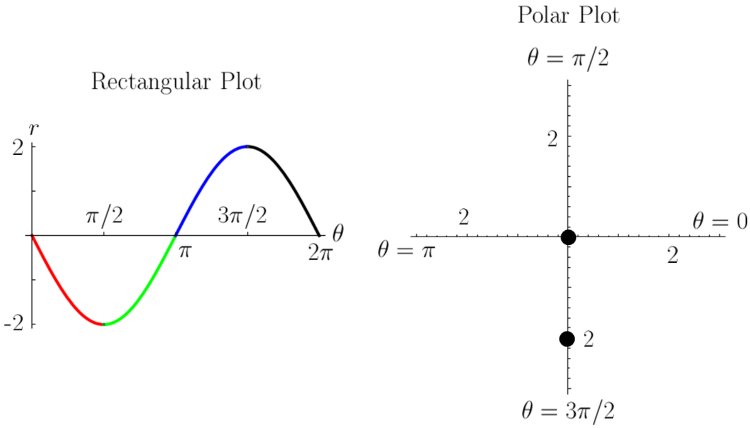
WORKED EXAMPLE: Plot the polar curve $r = -2\sin\theta$.





$\theta = \pi/2, r = -2 \implies$ March 2 units in **opposite direction** from $\theta = \pi/2$.

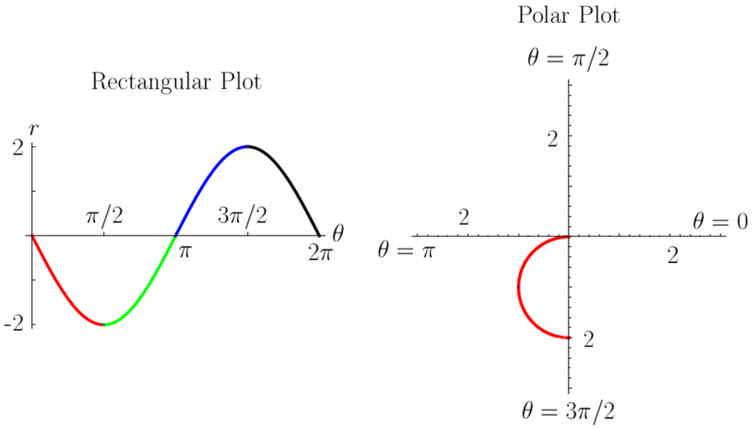
WORKED EXAMPLE: Plot the polar curve $r = -2\sin\theta$.



In Between: For $\theta \in (0, \pi/2) \subseteq$ QI, $r < 0$ and r **departs from zero**.

Polar Plots (Example)

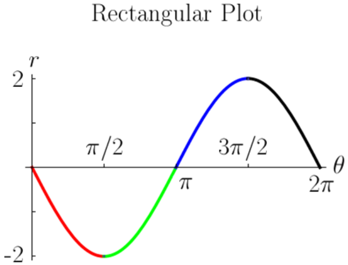
WORKED EXAMPLE: Plot the polar curve $r = -2\sin\theta$.

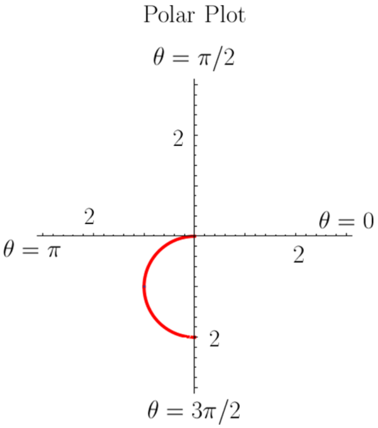


In Between: Trace a smooth curve in QIII that **departs from pole**.

Polar Plots (Example)

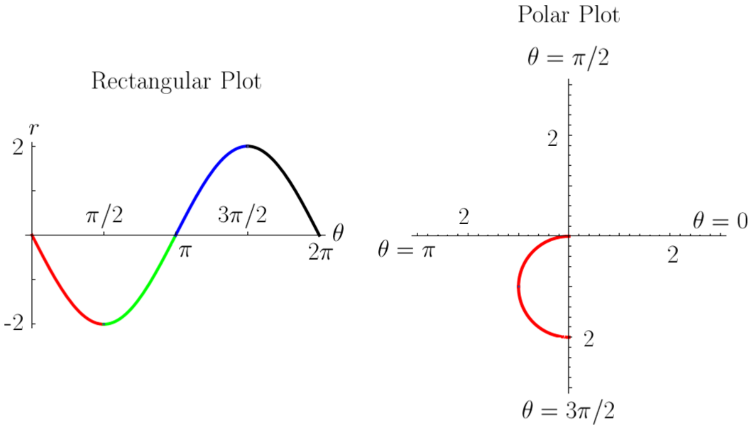
WORKED EXAMPLE: Plot the polar curve $r = -2\sin\theta$.





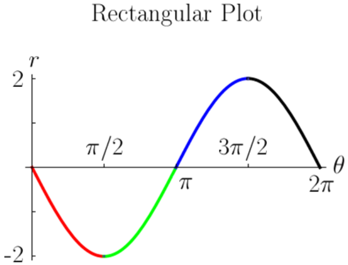
Consider the green portion of the rectangular plot (i.e. $\theta \in [\pi/2, \pi]$).

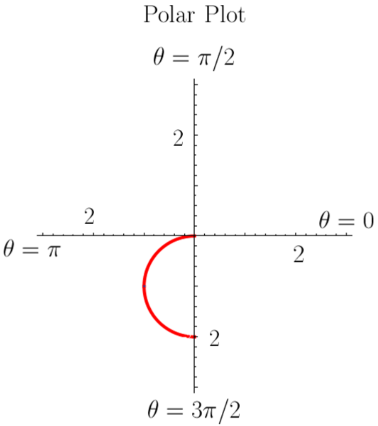
WORKED EXAMPLE: Plot the polar curve $r = -2\sin\theta$.



Starting Point: $\theta = \pi/2 \implies r = -2$.

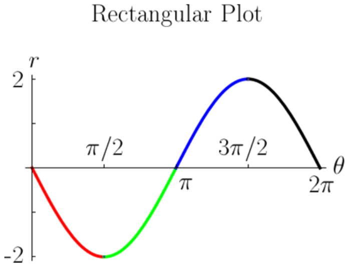
WORKED EXAMPLE: Plot the polar curve $r = -2\sin\theta$.

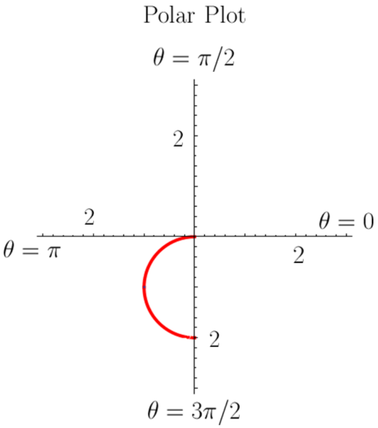




$\theta = \pi/2, r = -2 \implies$ March 2 units in **opposite direction** from $\theta = \pi/2$.

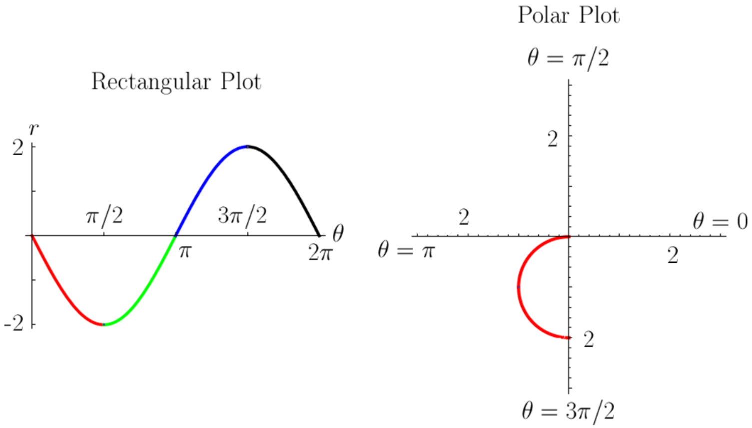
WORKED EXAMPLE: Plot the polar curve $r = -2\sin\theta$.





Ending Point: $\theta = \pi \implies r = 0$.

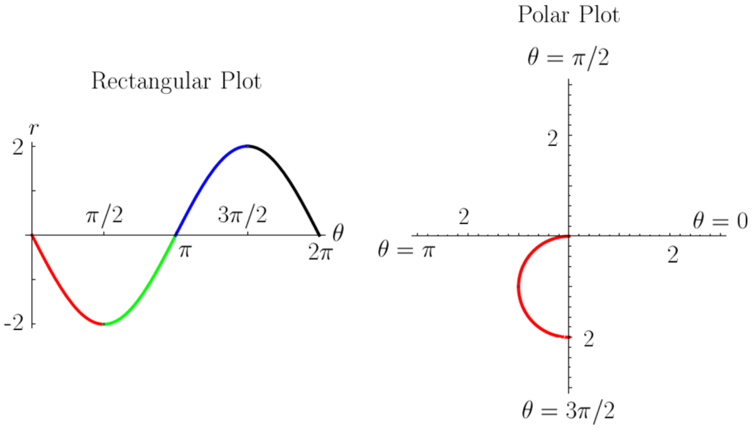
WORKED EXAMPLE: Plot the polar curve $r = -2\sin\theta$.



$$\theta = \pi, r = 0 \implies \textbf{pole}.$$

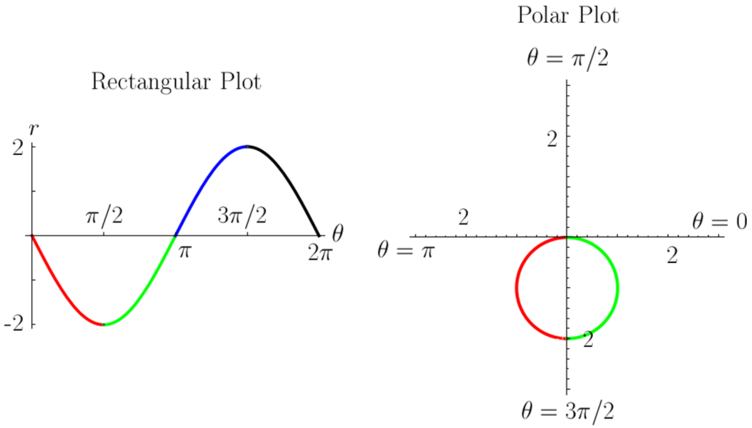
Polar Plots (Example)

WORKED EXAMPLE: Plot the polar curve $r = -2\sin\theta$.



In Between: For $\theta \in (\pi/2, \pi) =$ QII, $r < 0$ and r **approaches zero**.

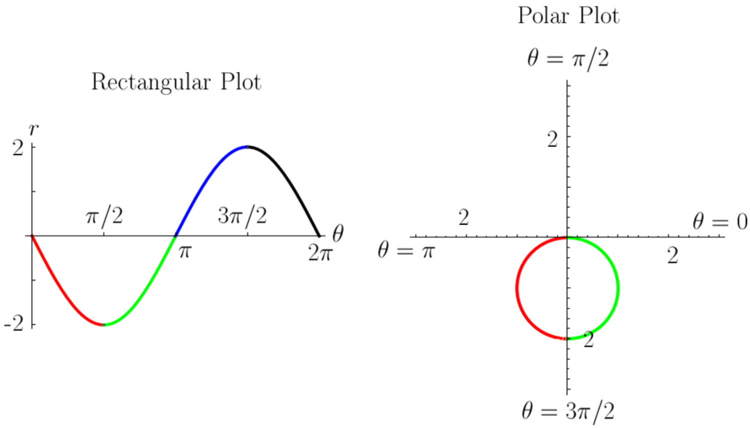
WORKED EXAMPLE: Plot the polar curve $r = -2\sin\theta$.



In Between: Trace a smooth curve in QIV that **approaches pole**.

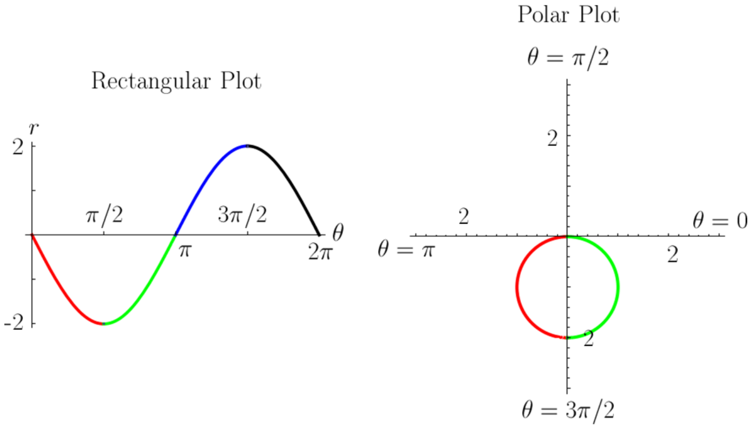
Polar Plots (Example)

WORKED EXAMPLE: Plot the polar curve $r = -2\sin\theta$.



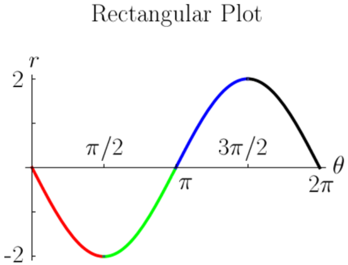
AND WE'RE DONE!

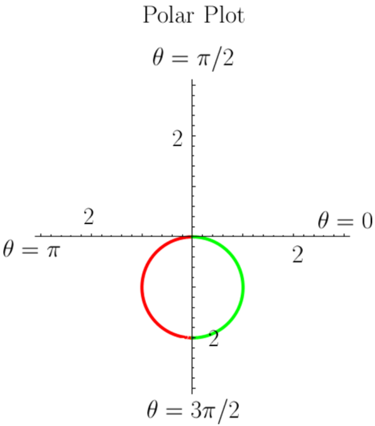
WORKED EXAMPLE: Plot the polar curve $r = -2\sin\theta$.



Done? Already? But why?? What about $\theta \in [\pi, 2\pi]$??

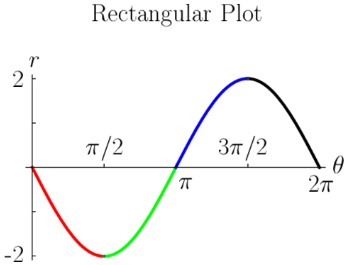
WORKED EXAMPLE: Plot the polar curve $r = -2\sin\theta$.

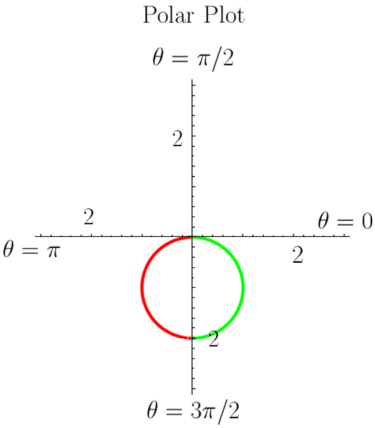




Because continuing further would **re-trace the same curve**!

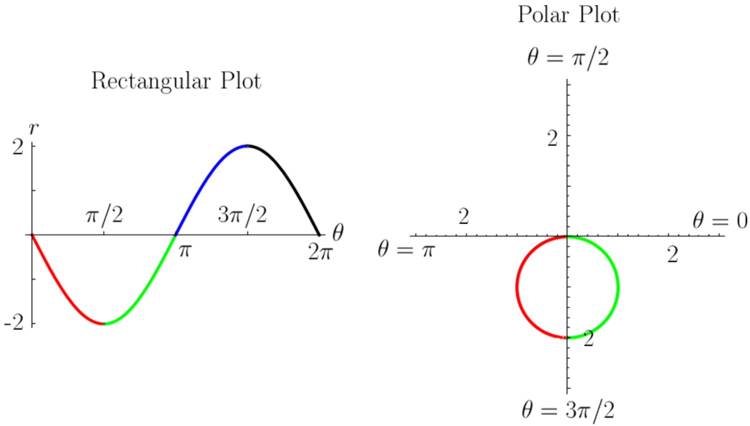
WORKED EXAMPLE: Plot the polar curve $r = -2\sin\theta$.





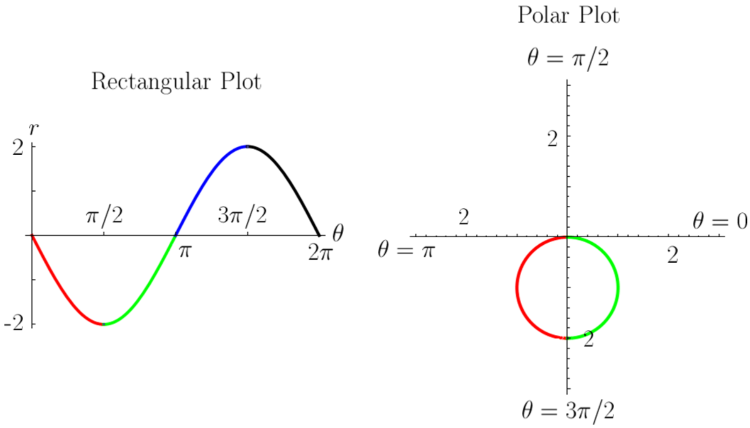
But let's continue anyway to see why the same curve is re-traced.....

WORKED EXAMPLE: Plot the polar curve $r = -2\sin\theta$.



Consider the blue portion of the rectangular plot (i.e. $\theta \in [\pi, 3\pi/2]$).

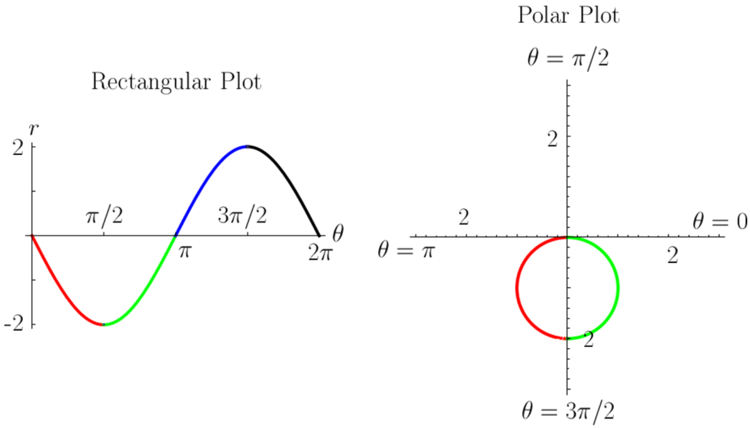
WORKED EXAMPLE: Plot the polar curve $r = -2\sin\theta$.



Starting Point: $\theta = \pi \implies r = 0$.

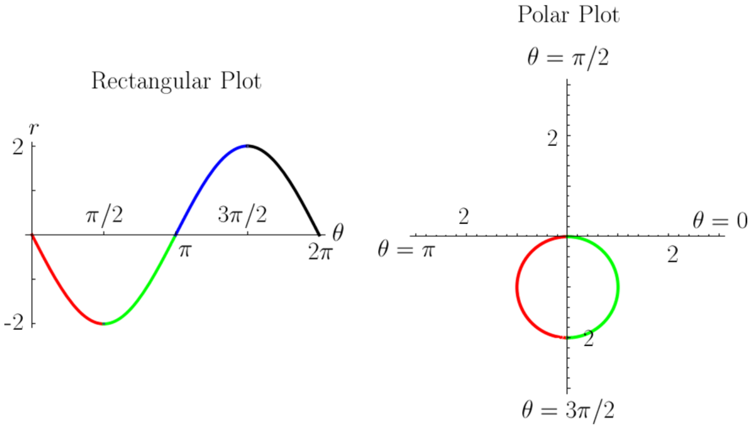
Polar Plots (Example)

WORKED EXAMPLE: Plot the polar curve $r = -2\sin\theta$.



$$\theta = \pi, r = 0 \implies \textbf{pole}.$$

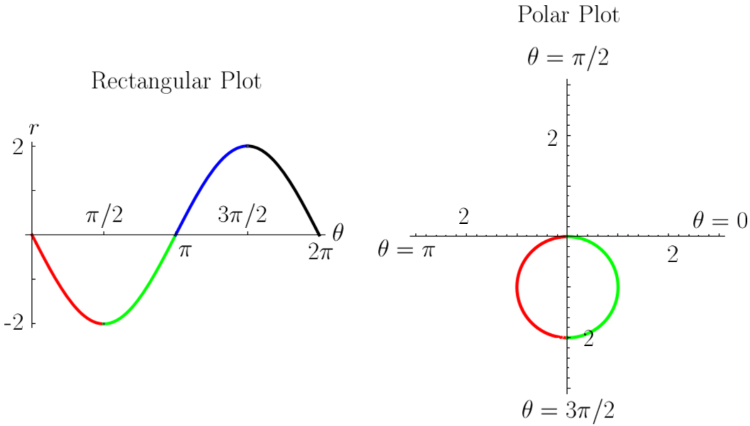
WORKED EXAMPLE: Plot the polar curve $r = -2\sin\theta$.



Ending Point: $\theta = 3\pi/2 \implies r = 2$.

Polar Plots (Example)

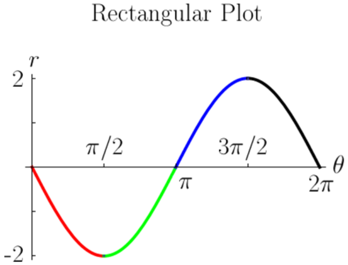
WORKED EXAMPLE: Plot the polar curve $r = -2\sin\theta$.

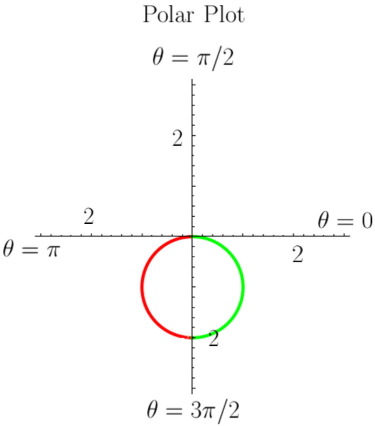


$\theta = 3\pi/2, r = 2 \implies$ March 2 units in **same direction** as $\theta = 3\pi/2$.

Polar Plots (Example)

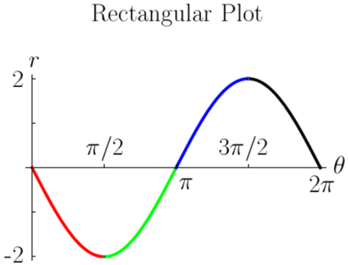
WORKED EXAMPLE: Plot the polar curve $r = -2\sin\theta$.

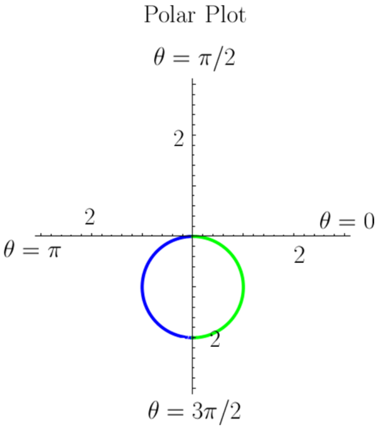




In Between: For $\theta \in (\pi, 3\pi/2) \subseteq$ QIII, $r > 0$ and r **departs from zero**.

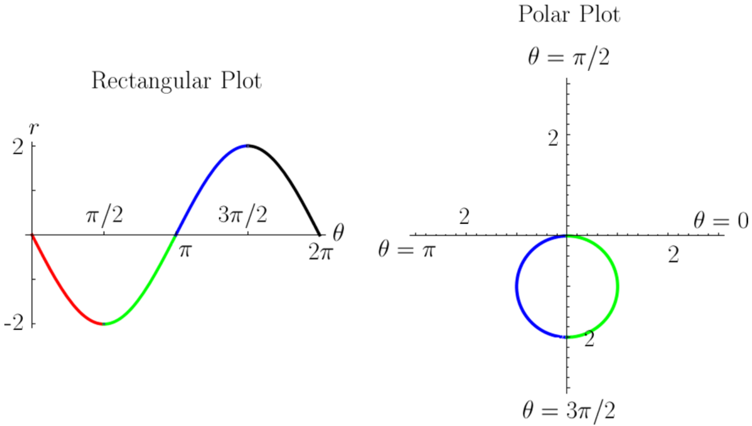
WORKED EXAMPLE: Plot the polar curve $r = -2\sin\theta$.





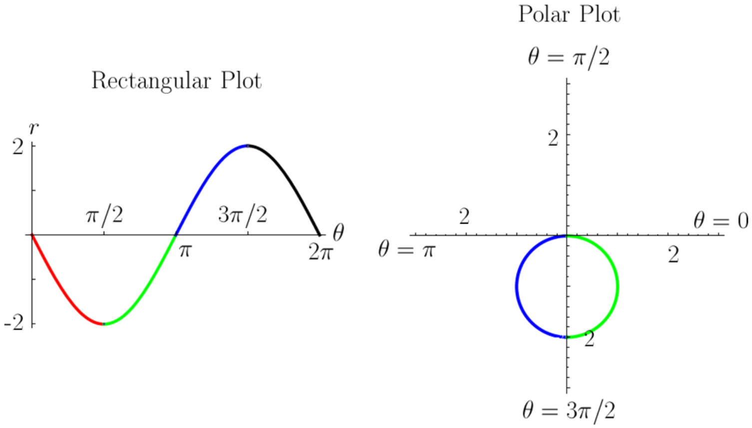
In Between: Trace a smooth curve in QIII that **departs from pole**.

WORKED EXAMPLE: Plot the polar curve $r = -2\sin\theta$.



Consider the black portion of the rectangular plot (i.e. $\theta \in [3\pi/2, 2\pi]$).

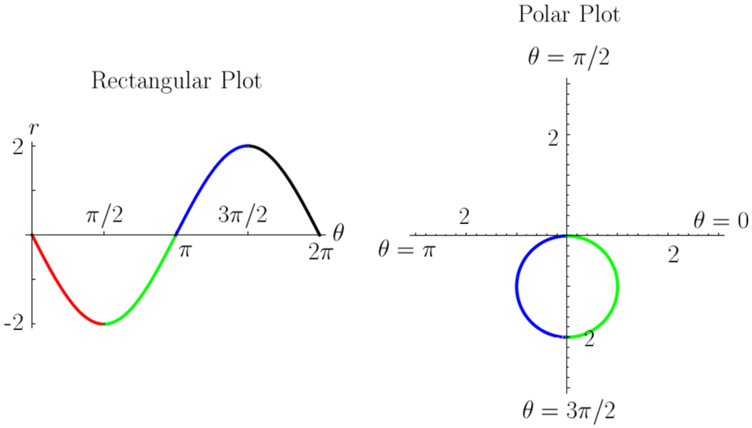
WORKED EXAMPLE: Plot the polar curve $r = -2\sin\theta$.



Starting Point: $\theta = 3\pi/2 \implies r = 2$.

Polar Plots (Example)

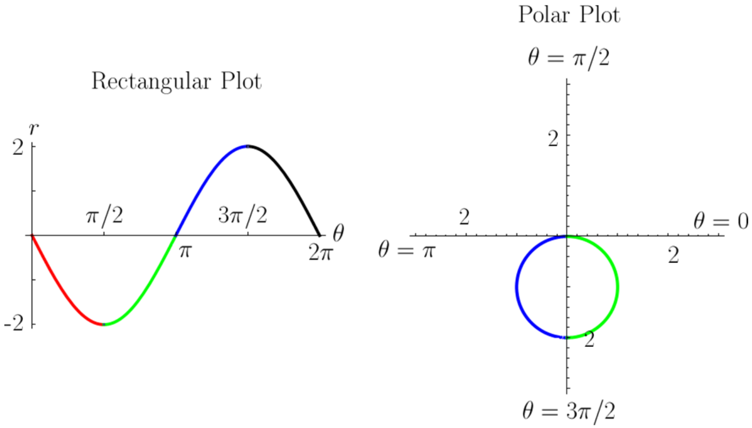
WORKED EXAMPLE: Plot the polar curve $r = -2\sin\theta$.



$\theta = 3\pi/2, r = 2 \implies$ March 2 units in **same direction** as $\theta = 3\pi/2$.

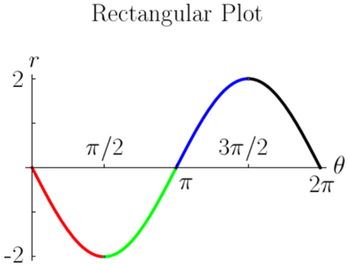
Polar Plots (Example)

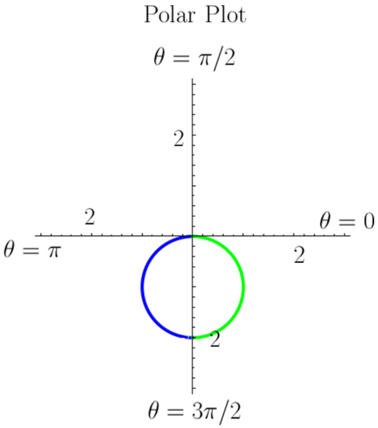
WORKED EXAMPLE: Plot the polar curve $r = -2\sin\theta$.



Ending Point: $\theta = 2\pi \implies r = 0$.

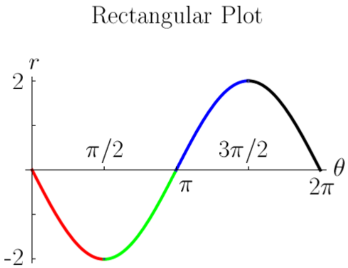
<u>WORKED EXAMPLE:</u> Plot the polar curve $r = -2\sin\theta$.

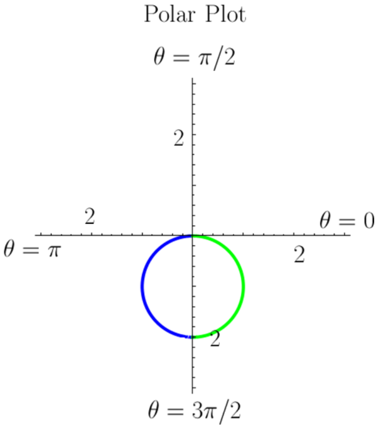




$$\theta = 2\pi, r = 0 \implies \textbf{pole}.$$

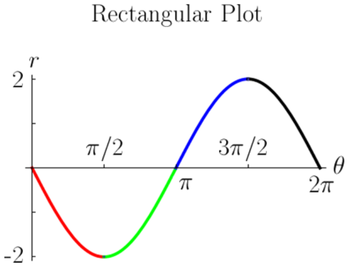
WORKED EXAMPLE: Plot the polar curve $r = -2\sin\theta$.

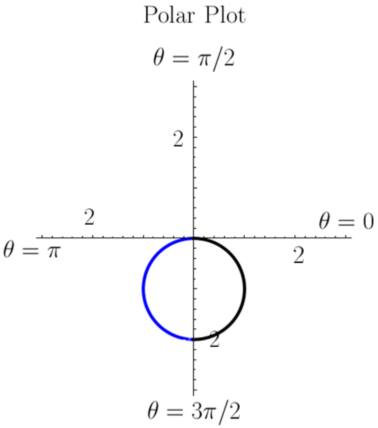




In Between: For $\theta \in (3\pi/2, 2\pi) \subseteq$ QIV, $r > 0$ and r **approaches zero**.

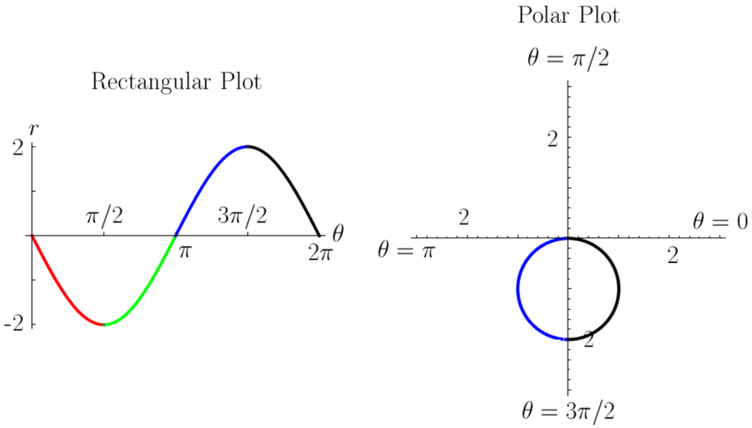
WORKED EXAMPLE: Plot the polar curve $r = -2\sin\theta$.





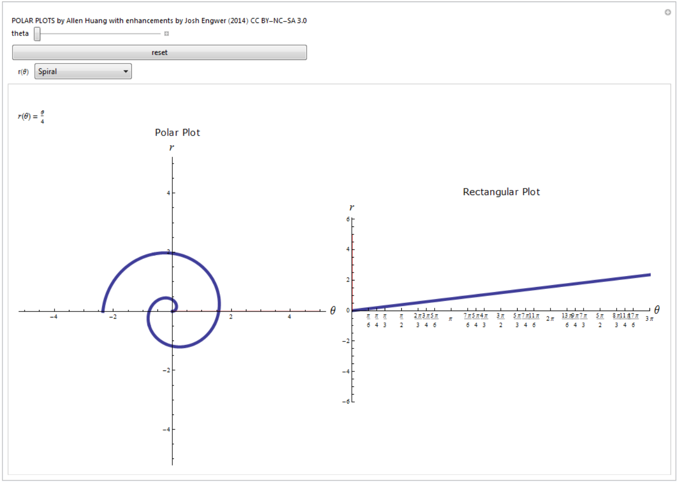
In Between: Trace a smooth curve in QIV that **approaches pole**.

WORKED EXAMPLE: Plot the polar curve $r = -2\sin\theta$.



Moral: NEVER RE-TRACE THE SAME POLAR CURVE!

(DEMO) POLAR PLOTS (Click below):

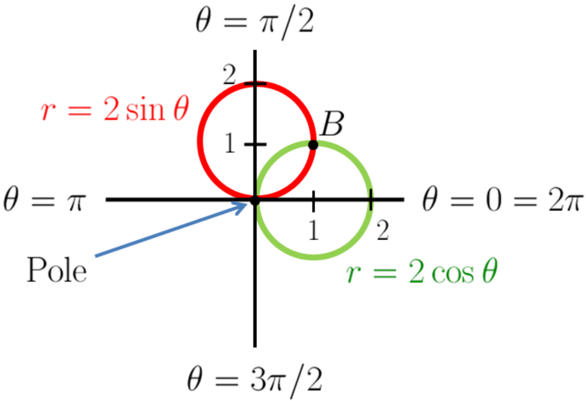


Intersection of Two Polar Curves

TASK: Find all intersection points of the polar curves $r = f(\theta)$ & $r = g(\theta)$.

- Solving $f(\theta) = g(\theta)$ finds some, but not necessarily all, intersection points.
- In particular, intersections at the **pole** are nearly impossible to find algebraically because the pole has no single representation in polar coordinates that satisfies both $r = f(\theta)$ & $r = g(\theta)$.
- Therefore, to find all intersection points, **graph both curves**!

Intersection of Two Polar Curves (Example)

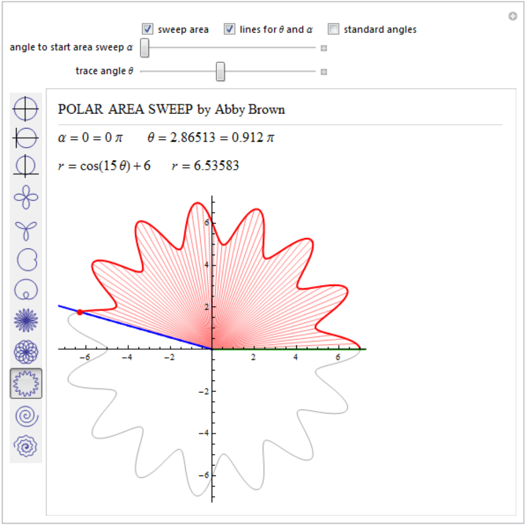


$2\sin\theta = 2\cos\theta \implies \cos\theta(\tan\theta - 1) = 0 \implies \cos\theta = 0$ or $\tan\theta = 1$
$\implies \theta \in \{\pi/2, 3\pi/2\}$ or $\theta \in \{\pi/4, 5\pi/4\}$
Discard $\theta = \pi/2$ & $\theta = 3\pi/2$ since they're **extraneous solutions**.
Both $\theta = \pi/4$ and $\theta = 5\pi/4$ yield the same intersection point B.
So solving algebraically did not yield the intersection point at the pole.

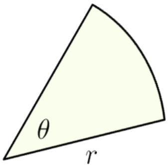
POLAR CURVES PART II:
AREAS OF POLAR REGIONS

Polar Regions (Demo)

(DEMO) POLAR REGIONS (Click below):



Area of a Sector

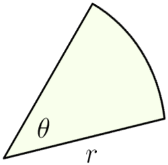


Proposition

The area of a sector with radius r that sweeps an angle θ is:

$$\textit{Area of Sector}(r, \theta) = \frac{1}{2} r^2 \theta$$

Area of a Sector



Proposition

The area of a sector with radius r that sweeps an angle θ is:

$$\textit{Area of Sector}(r,\theta) = \frac{1}{2}r^2\theta$$

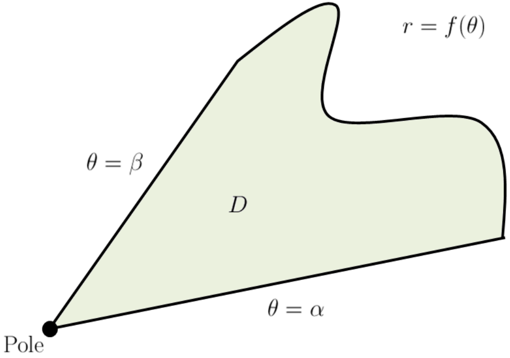
<u>PROOF:</u> Area of Circle$(r) = \pi r^2$.
But a circle is just a sector that sweeps an angle of 2π radians.

$$\implies \text{Area of Sector}(r,\theta) = \left(\frac{\theta}{2\pi}\right)\left[\text{Area of Circle}(r)\right] = \left(\frac{\theta}{2\pi}\right)\left(\pi r^2\right) = \frac{1}{2}r^2\theta$$

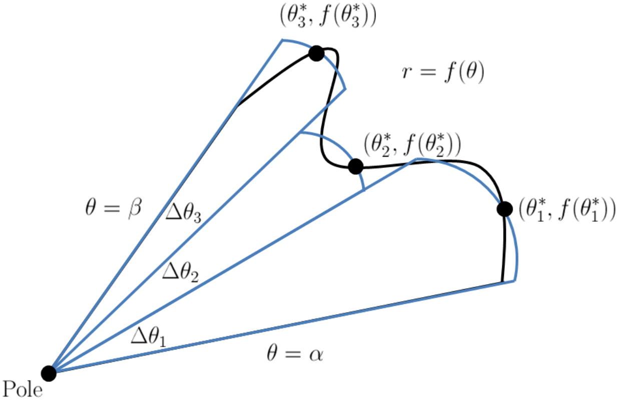
QED

Area of a Polar Region bounded by One Polar Curve

Let D be the region bounded by polar curve $r = f(\theta)$ and rays $\theta = \alpha$, $\theta = \beta$.



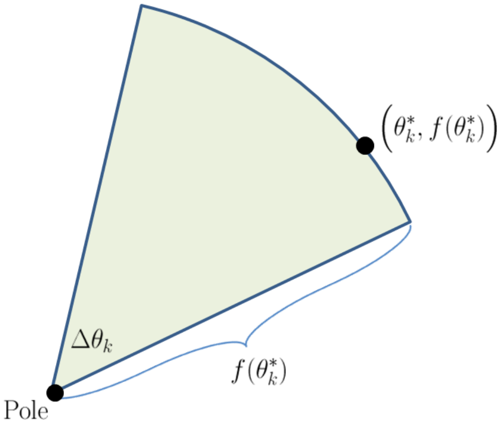
Area of a Polar Region bounded by One Polar Curve



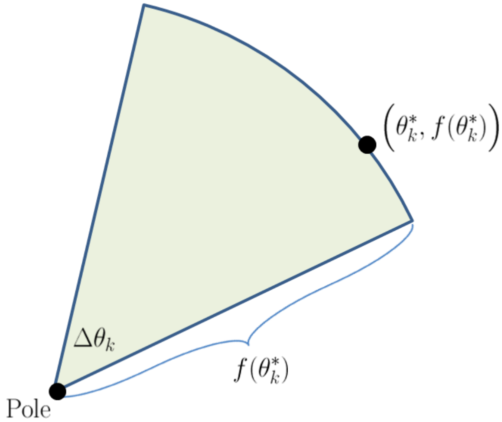
Subdivide polar region into N **sectors** and with **tags** $\theta_1^*, \theta_2^*, \ldots$.

Area of a Polar Region bounded by One Polar Curve

Consider the k^{th} Sector:



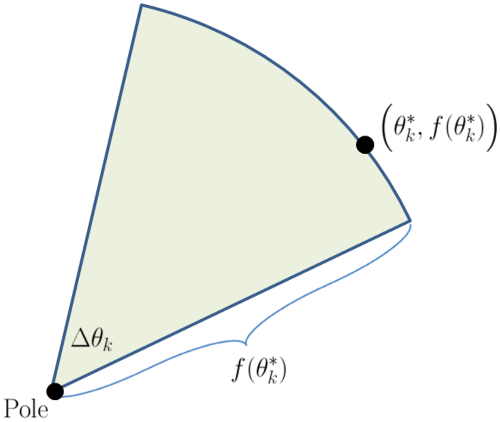
Area of a Polar Region bounded by One Polar Curve



k^{th} Sector in D:

Angle	$=$	$\Delta\theta_k$
Radius	$=$	(Distance from Pole to Polar Curve)
Area	$=$	$\frac{1}{2} \times (\text{Radius})^2 \times (\text{Angle})$

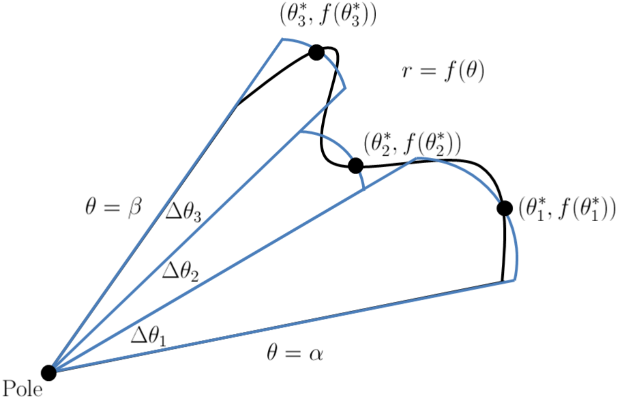
Area of a Polar Region bounded by One Polar Curve



k^{th} Sector in D:

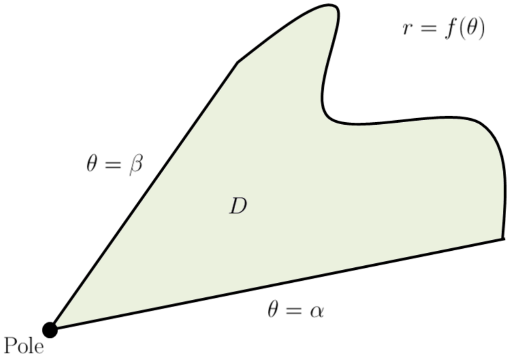
$$\begin{array}{lcl} \text{Angle} & = & \Delta\theta_k \\ \text{Radius} & = & |f(\theta_k^*)| \\ \hline \text{Area} & = & \frac{1}{2}\left[f(\theta_k^*)\right]^2 \Delta\theta_k \end{array}$$

Area of a Polar Region bounded by One Polar Curve



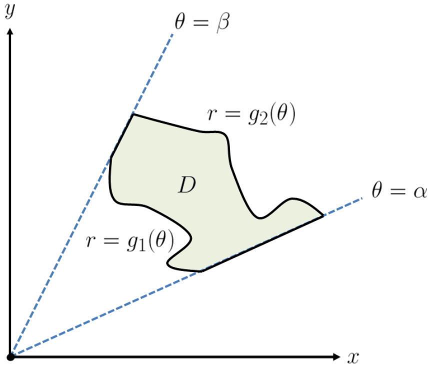
Riemann Sum: Area$(D) \approx A_N^* := \sum_{k=1}^{N} \frac{1}{2} [f(\theta_k^*)]^2 \Delta\theta_k$

Area of a Polar Region bounded by One Polar Curve



Integral: $\text{Area}(D) = \lim_{N\to\infty} A_N^* = \int_{\text{smallest } \theta\text{-value in } D}^{\text{largest } \theta\text{-value in } D} \frac{1}{2}\left[f(\theta)\right]^2 \, d\theta = \int_{\alpha}^{\beta} \frac{1}{2}\left[f(\theta)\right]^2 \, d\theta$

Radially-Simple (r-Simple) Polar Regions (Definition)



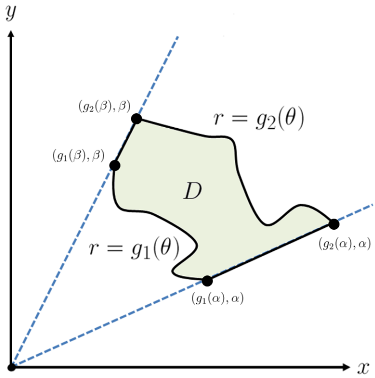
Above, the **inner BC** is $r = g_1(\theta)$ & the **outer BC** is $r = g_2(\theta)$.
Notice, the **inner BC** & **outer BC** are both traced for $\theta \in [\alpha, \beta]$.

Definition

A region $D \subset \mathbb{R}^2$ is **radially-simple (r-Simple)** if

D has one **inner BC** & one **outer BC**, both traced over the <u>same</u> θ-values.

Radially-Simple (r-Simple) Polar Regions (Definition)



Above, the **inner BC** is $r = g_1(\theta)$ & the **outer BC** is $r = g_2(\theta)$.
Notice, the **inner BC** & **outer BC** are both traced for $\theta \in [\alpha, \beta]$.

Definition

A region $D \subset \mathbb{R}^2$ is **radially-simple (r-Simple)** if

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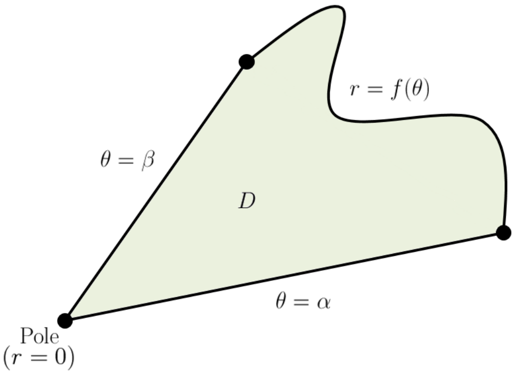
Radially-Simple (r-Simple) Polar Regions (Definition)



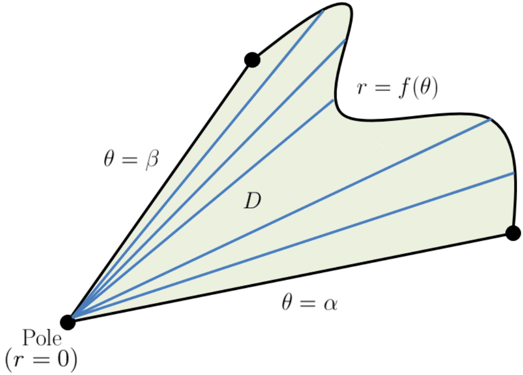
Above, the **inner BC** is the **pole** ($r = 0$) & the **outer BC** is $r = f(\theta)$.
Notice, the **outer BC** is traced for $\theta \in [\alpha, \beta]$.

Definition

A region $D \subset \mathbb{R}^2$ is **radially-simple (r-Simple)** if

D has the **pole** as the **inner BC** & only one **outer BC**.

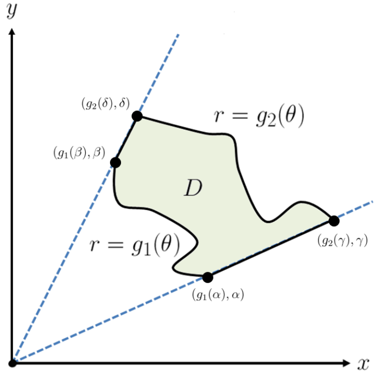
Radially-Simple (r-Simple) Polar Regions (Definition)



Above, the **inner BC** is the **pole** $(r = 0)$ & the **outer BC** is $r = f(\theta)$.

i.e., r-Simple regions can be swept radially from the pole (with rays [in **blue**]) where each ray enters the **same inner BC** & exits the **same outer BC**.

Quasi-Radially-Simple (Quasi-r-Simple) Polar Region



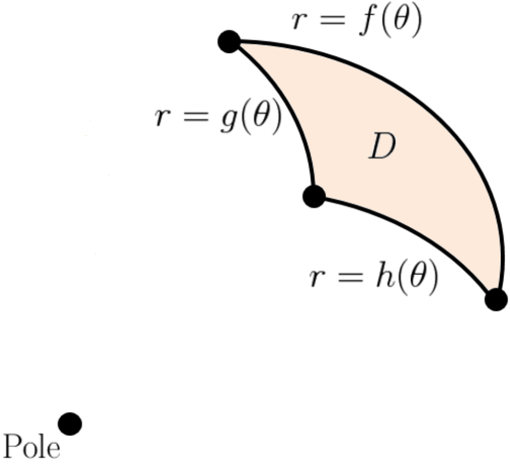
Above, the **inner BC** is $r = g_1(\theta)$ & the **outer BC** is $r = g_2(\theta)$.
Notice, **inner BC** is traced for $\theta \in [\alpha, \beta]$ & **outer BC** is traced for $\theta \in [\gamma, \delta]$.

Definition

A region $D \subset \mathbb{R}^2$ is **quasi-radially-simple (Quasi-r-Simple)** if

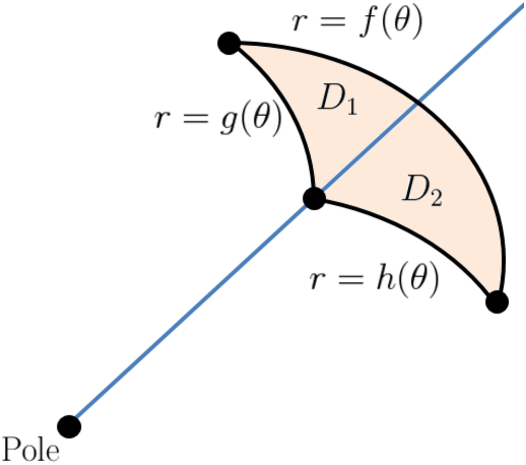
D has one **inner BC** & one **outer BC**, each traced over different θ-values.

Region that's neither r-Simple nor Quasi-r-Simple



How to handle a polar region that lacks (quasi-) radial simplicity???

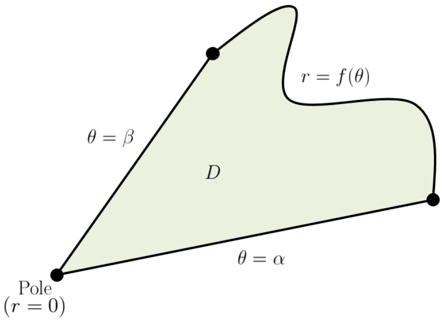
Region that's neither r-Simple nor Quasi-r-Simple



Subdivide polar region **radially from the pole** through appropriate BP.

Then, $D = D_1 \cup D_2 \implies \text{Area}(D) = \text{Area}(D_1) + \text{Area}(D_2)$

Area of a Radially Simple (r-Simple) Polar Region

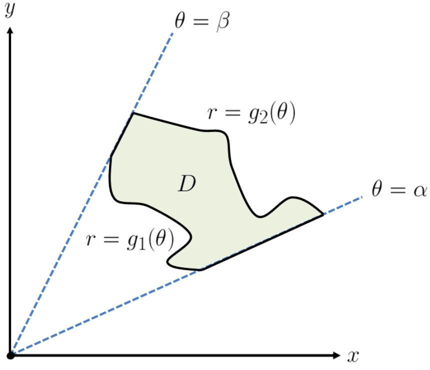


Proposition

Let D be a r-simple region as shown above. Then:

$$\text{Area}(D) = \int_{\text{Smallest } \theta\text{-value in } D}^{\text{Largest } \theta\text{-value in } D} \frac{1}{2}(\text{Outer BC of } D)^2 \, d\theta = \int_{\alpha}^{\beta} \frac{1}{2}[f(\theta)]^2 \, d\theta$$

Area of a Radially Simple (r-Simple) Polar Region

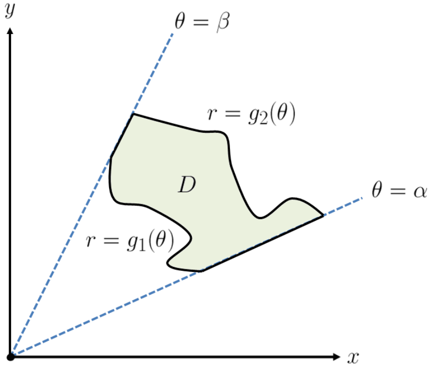


Proposition

Let D be a r-simple region as shown above. Then:

$$\textit{Area}(D) = \frac{1}{2}\int_{\textit{Smallest } \theta\textit{-value in } D}^{\textit{Largest } \theta\textit{-value in } D} (\textit{Outer BC of } D)^2 - (\textit{Inner BC of } D)^2 \, d\theta$$

Area of a Radially Simple (r-Simple) Polar Region

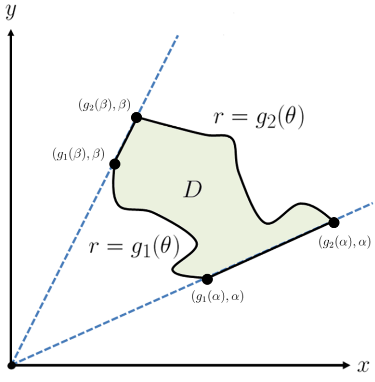


Proposition

Let D be a r-simple region as shown above. Then:

$$\text{Area}(D) = \frac{1}{2}\int_{\alpha}^{\beta} [g_2(\theta)]^2 - [g_1(\theta)]^2 \, d\theta$$

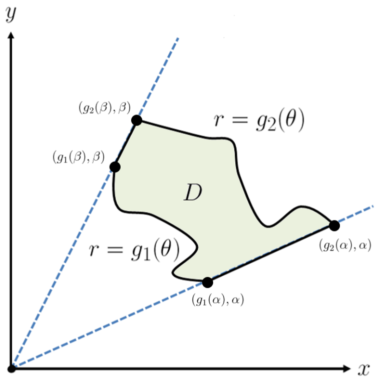
Area of a Radially Simple (r-Simple) Polar Region



Proposition

Let D be a r-simple region as shown above. Then:

$$\text{Area}(D) = \frac{1}{2} \int_{\text{Smallest } \theta\text{-value in } D}^{\text{Largest } \theta\text{-value in } D} (\text{Outer BC of } D)^2 - (\text{Inner BC of } D)^2 \, d\theta$$

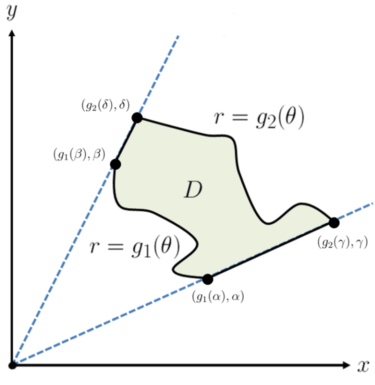


Proposition

Let D be a r-simple region as shown above. Then:

$$\textit{Area}(D) = \frac{1}{2}\int_{\alpha}^{\beta} [g_2(\theta)]^2 - [g_1(\theta)]^2 \, d\theta$$

Area of a Quasi-r-Simple Polar Region

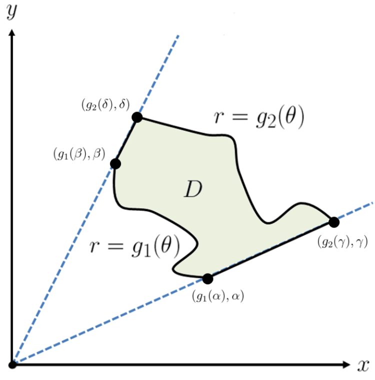


Proposition

Let D be a quasi-r-simple region as shown above. Then Area(D) =

$$\frac{1}{2}\int_{\text{Smallest }\theta\text{ for Outer BC}}^{\text{Largest }\theta\text{ for Outer BC}} (\text{Outer BC})^2\, d\theta - \frac{1}{2}\int_{\text{Smallest }\theta\text{ for Inner BC}}^{\text{Largest }\theta\text{ for Inner BC}} (\text{Inner BC})^2\, d\theta$$

Area of a Quasi-r-Simple Polar Region



Proposition

Let D be a quasi-r-simple region as shown above. Then:

$$\textit{Area}(D) = \frac{1}{2}\int_{\gamma}^{\delta} [g_2(\theta)]^2 \, d\theta - \frac{1}{2}\int_{\alpha}^{\beta} [g_1(\theta)]^2 \, d\theta$$

Fin

Fin.