Arc Length & Surface Area

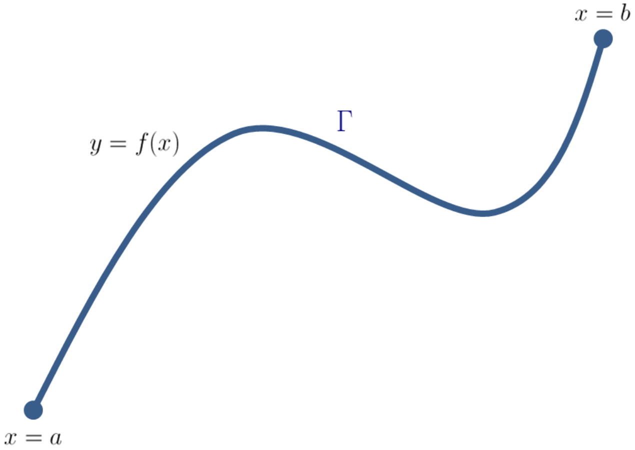
Calculus II

Josh Engwer

TTU

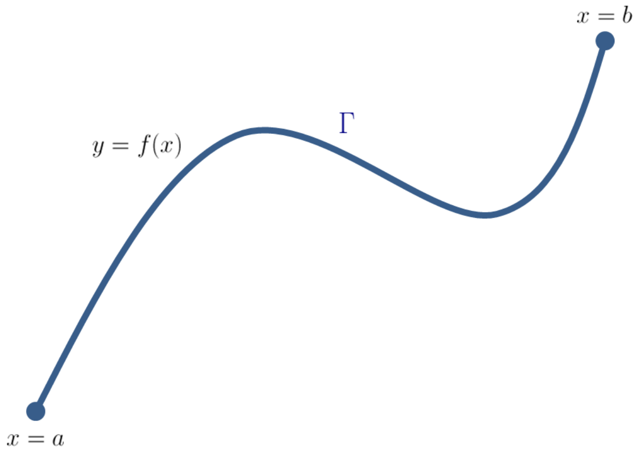
10 February 2014

Arc Length of a Curve



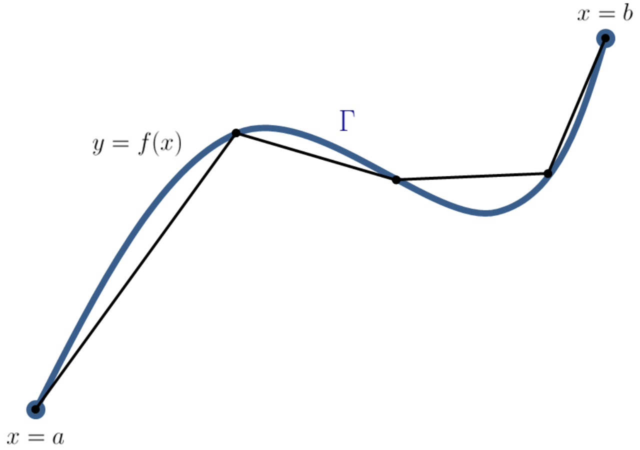
Let Γ be the portion of curve $f(x)$ for $x \in [a, b]$

Arc Length of a Curve



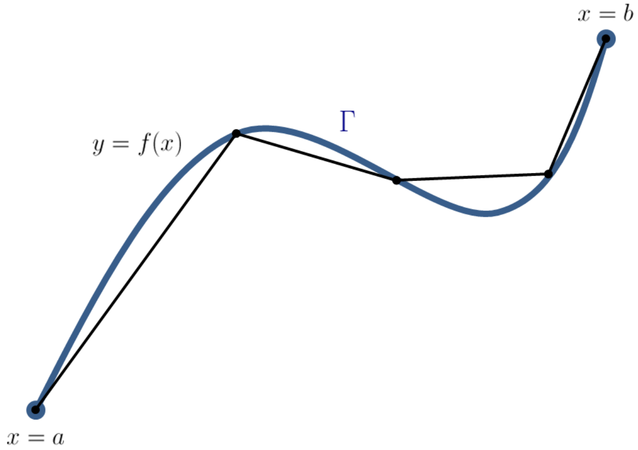
TASK: Find ArcLength(Γ)

Arc Length of a Curve



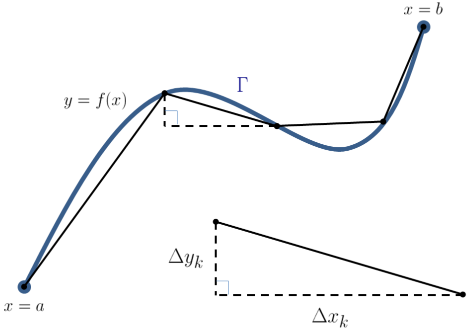
Partition curve Γ into N subarcs & line segments

Arc Length of a Curve



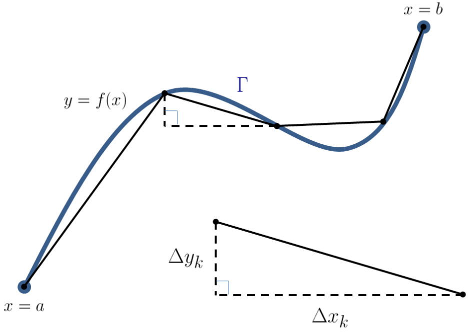
Key Element: **Line Segment**

Arc Length of a Curve



$$k^{th} \text{ Line Segment on } \Gamma: \quad \begin{array}{lcl} \text{Width} & = & \Delta x_k \\ \text{Height} & = & \Delta y_k \\ \hline \text{Length} & = & \sqrt{(\text{Width})^2 + (\text{Height})^2} \end{array}$$

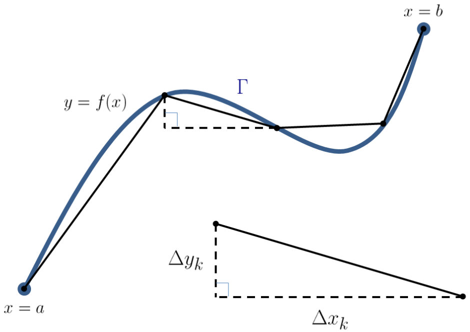
Arc Length of a Curve



k^{th} Line Segment on Γ:

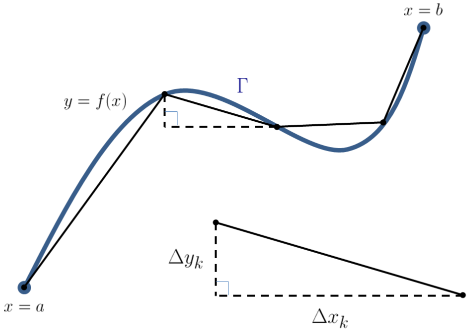
Width	$=$	Δx_k
Height	$=$	Δy_k
Length	$=$	$\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$

Arc Length of a Curve



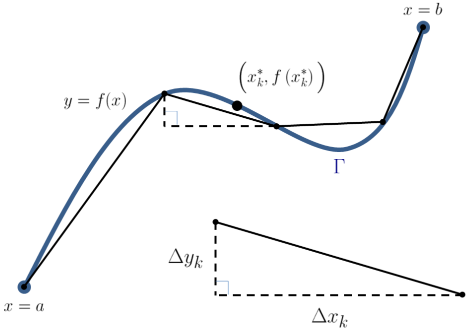
k^{th} Line Segment on Γ: Length $= \sqrt{(\Delta x_k)^2 \left[1 + \frac{(\Delta y_k)^2}{(\Delta x_k)^2}\right]}$

Arc Length of a Curve



k^{th} Line Segment on Γ: Length $= \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2}\Delta x_k$

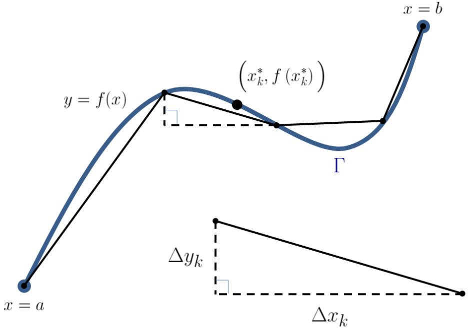
Arc Length of a Curve



k^{th} Line Segment on Γ:

$$\text{Length} \quad = \quad \sqrt{1 + \left(\frac{\Delta y_k}{\Delta x_k}\right)^2}\Delta x_k \stackrel{MVT}{=} \sqrt{1 + \left[f'\left(x_k^*\right)\right]^2}\Delta x_k$$

Arc Length of a Curve



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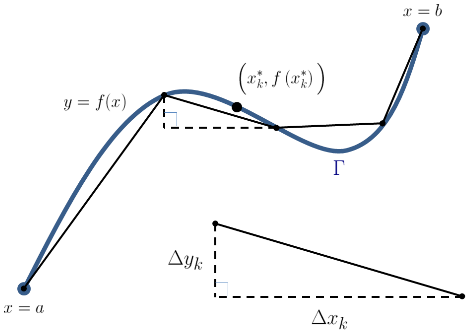

Riemann Sum: ArcLength$(\Gamma) \approx L_N^* := \sum_{k=1}^{N} \sqrt{1 + \left[f'\left(x_k^*\right)\right]^2}\Delta x_k$

Integral: ArcLength$(\Gamma) = \lim_{N\to\infty} L_N^* = \int_a^b \sqrt{1 + \left[f'(x)\right]^2}\, dx$

Arc Length of a Curve

Proposition

Let function $f \in C^1[a,b]$.
Let Γ *be the portion of the curve* $y = f(x)$ *for* $x \in [a,b]$. *Then:*

$$\textit{ArcLength}(\Gamma) = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

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Let function $g \in C^1[c,d]$.
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GOOD NEWS: Don't bother with characterizing the k^{th} line segment.
Just go straight to the integral. (i.e. memorize the above integral forms)
You can do this because no region is involved (only a curve).

Arc Length of a Polar Curve

Since the Arc Length of a Curve in Rectangular Coordinates has already been established, there's no need to start from "first principles":

Let Γ be the polar curve $r = f(\theta)$ bounded by the rays $\theta = \alpha$ & $\theta = \beta$.

k^{th} Line Segment on Γ:

$$\begin{array}{lcl} \text{Width} & = & \Delta x_k \\ \text{Height} & = & \Delta y_k \\ \hline \text{Length} & = & \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} \end{array}$$

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Integral: ArcLength$(\Gamma) = \lim_{N\to\infty} L_N^* = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$

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$$\begin{cases} x = r\cos\theta = f(\theta)\cos\theta \\ y = r\sin\theta = f(\theta)\sin\theta \end{cases}$$

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$$\begin{cases} x = r\cos\theta = f(\theta)\cos\theta \\ y = r\sin\theta = f(\theta)\sin\theta \end{cases} \implies \begin{cases} \dfrac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta \\ \dfrac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta \end{cases}$$

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$$\begin{cases} \dfrac{dx}{d\theta} = f'(\theta)\cos\theta - f(\theta)\sin\theta \\ \dfrac{dy}{d\theta} = f'(\theta)\sin\theta + f(\theta)\cos\theta \end{cases} \implies \left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = [f(\theta)]^2 + [f'(\theta)]^2$$

Arc Length of a Polar Curve

Proposition

Let function $f \in C^1[\alpha, \beta]$.
Let Γ *be the polar curve* $r = f(\theta)$ *bounded by the rays* $\theta = \alpha$ *&* $\theta = \beta$. *Then:*

$$\textit{ArcLength}(\Gamma) = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_{\alpha}^{\beta} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta$$

Arc Length of a Polar Curve

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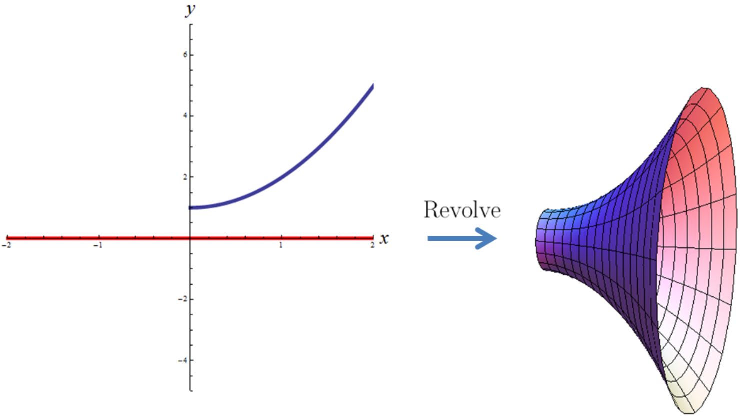
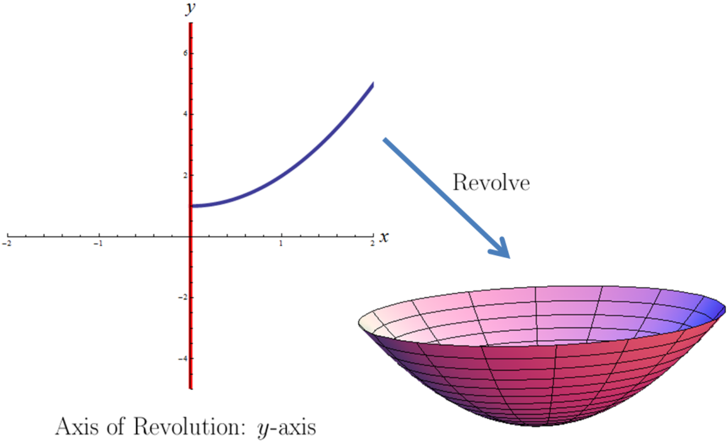
Surfaces of Revolution

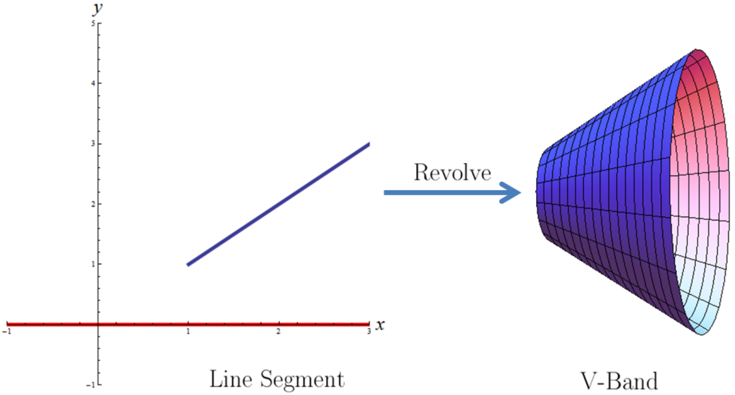


Axis of Revolution: x-axis

Surfaces of Revolution

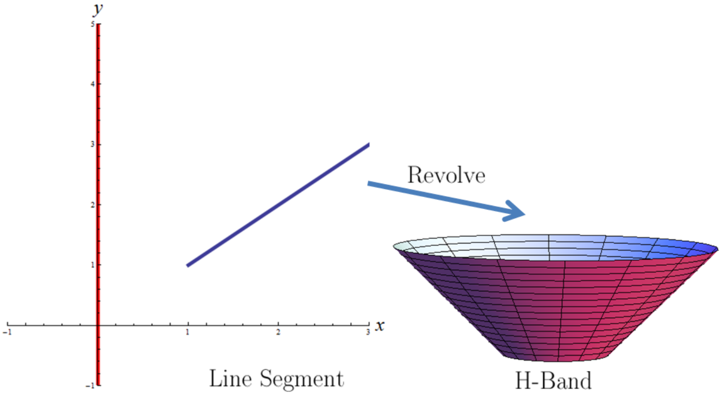


Surfaces of Revolution (V-Bands)



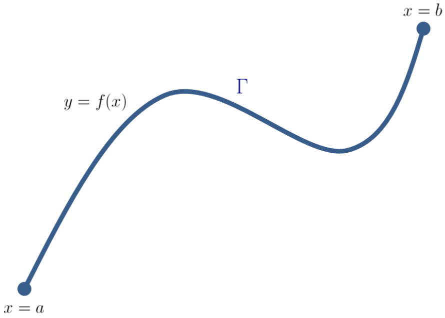
Axis of Revolution: x-axis

Surfaces of Revolution (H-Bands)



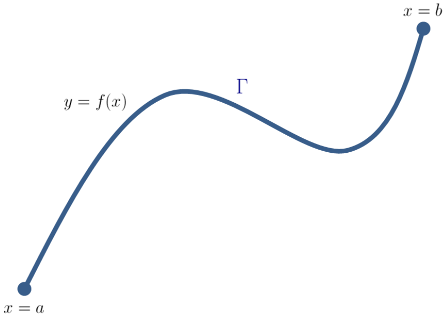
Axis of Revolution: y-axis

Surface Area of Surfaces Revolved about x-Axis



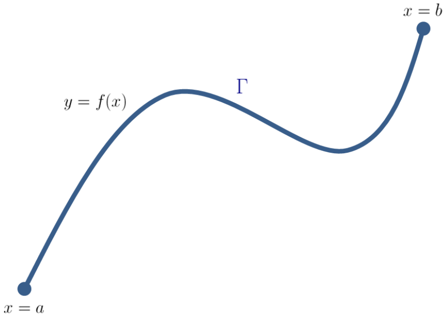
Let Γ be the portion of the curve $f(x)$ for $x \in [a, b]$.

Surface Area of Surfaces Revolved about x-Axis



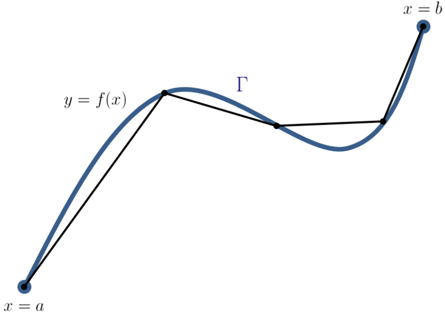
Let S be the surface formed by revolving Γ about the x-axis.

Surface Area of Surfaces Revolved about x-Axis



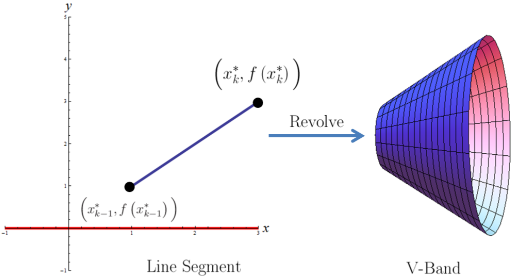
TASK: Find SurfaceArea(S)

Surface Area of Surfaces Revolved about x-Axis



Partition curve Γ into N subarcs & line segments.

Surface Area of Surfaces Revolved about x-Axis

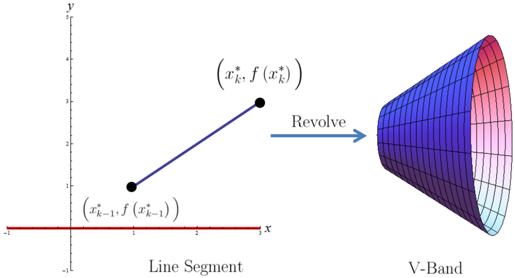


Axis of Revolution: x-axis

k^{th} V-Band on S:

$$\begin{array}{lcl} \text{Average Radius} & = & \frac{1}{2}\Big[(\text{Smallest Radius}) + (\text{Largest Radius})\Big] \\ \text{Slant Height} & = & (\text{Length of generating Line Segment}) \\ \hline \text{Surface Area} & = & 2\pi \times (\text{Average Radius}) \times (\text{Slant Height}) \end{array}$$

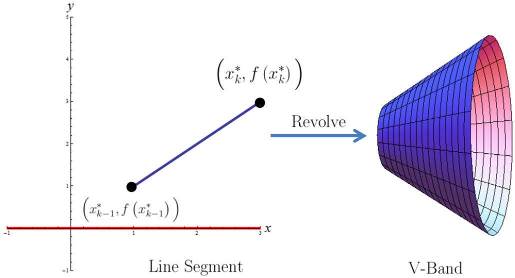
Surface Area of Surfaces Revolved about x-Axis



Axis of Revolution: x-axis

k^{th} V-Band on S:

$$
\begin{array}{lcl}
\text{Average Radius} & = & \frac{1}{2}\left[f\left(x_{k-1}^*\right) + f\left(x_k^*\right)\right] \\
\text{Slant Height} & = & \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} \\
\hline
\text{Surface Area} & = & 2\pi \times (\text{Average Radius}) \times (\text{Slant Height})
\end{array}
$$



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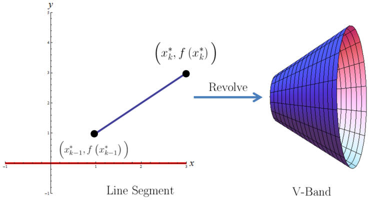
$$\begin{array}{llll} \text{Average Radius} & = \frac{1}{2}\left[f\left(x_{k-1}^*\right) + f\left(x_k^*\right)\right] & \approx & f\left(x_k^*\right) \\ \text{Slant Height} & = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} & \overset{MVT}{=} & \sqrt{1 + \left[f'\left(x_k^*\right)\right]^2}\,\Delta x_k \\ \hline \text{Surface Area} & \approx 2\pi f\left(x_k^*\right)\sqrt{1 + \left[f'\left(x_k^*\right)\right]^2}\,\Delta x_k & & \end{array}$$

Surface Area of Surfaces Revolved about x-Axis



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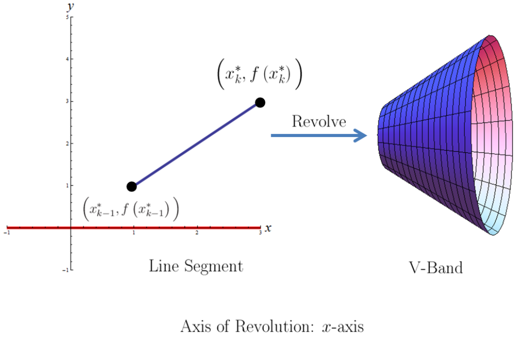
$$\begin{array}{lcl} \text{Average Radius} & = & \frac{1}{2}\left[f\left(x_{k-1}^*\right)+f\left(x_k^*\right)\right] \approx f\left(x_k^*\right) \\ \text{Slant Height} & = & \sqrt{(\Delta x_k)^2+(\Delta y_k)^2} \overset{MVT}{=} \sqrt{1+\left[f'(x_k^*)\right]^2}\Delta x_k \\ \hline \text{Surface Area} & \approx & 2\pi f\left(x_k^*\right)\sqrt{1+\left[f'(x_k^*)\right]^2}\Delta x_k \end{array}$$

Riemann Sum: SurfaceArea$(S) \approx SA_N^* := \sum_{k=1}^{N} 2\pi f\left(x_k^*\right)\sqrt{1+\left[f'(x_k^*)\right]^2}\Delta x_k$

Surface Area of Surfaces Revolved about x-Axis



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Surface Area of Surfaces of Revolution

Proposition

Let function $f \in C^1[a, b]$.
Let Γ *be the portion of the curve* $y = f(x)$ *for* $x \in [a, b]$.
Let S *be the surface formed by revolving* Γ *about the* x***-axis**. Then:*

$$\textit{SurfaceArea}(S) = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx$$

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Let function $g \in C^1[c, d]$.
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GOOD NEWS: Just go straight to the integral.

Surface Area of Polar Surfaces of Revolution

Proposition

Let function $f \in C^1[\alpha, \beta]$.
Let Γ be the portion of the curve $r = f(\theta)$ bounded by rays $\theta = \alpha$ & $\theta = \beta$.
*Let S be the surface formed by revolving Γ about the x-**axis**. Then:*

$$\textit{SurfaceArea}(S) = \int_\alpha^\beta 2\pi f(\theta) \sin\theta \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta$$

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Fin.