Physics Applications

Calculus II

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TTU

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PHYSICS PART I:

BASICS OF WORK, SPRINGS

Units of Measure in Physics

Be aware of the units of measure in work problems:

Mass	**Distance**	**Force**	**Work**
kg	m	N	J
g	cm	dyne	erg
slug	ft	lb	ft-lb

Work Done by a **Constant Force** along a Straight Line

Definition

A **force** on an object is a push or a pull applied to the object.

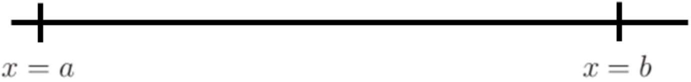
REMARK: The **weight** of an object is the (downward) force of **gravity** acting on the object.

Proposition

*The **work** done by a constant force on an object moving it a constant distance along a straight line is defined to be:*

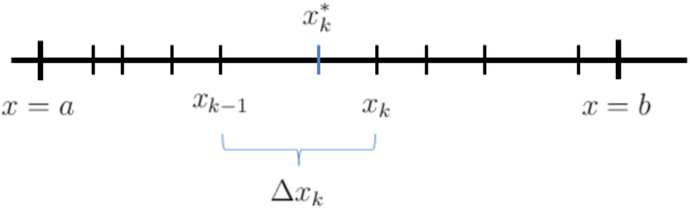
$$\Big(\mathit{Work}\Big) = \Big(\mathit{Force}\Big) \times \Big(\mathit{Distance}\Big)$$

Work Done by a **Variable Force** along the x-axis



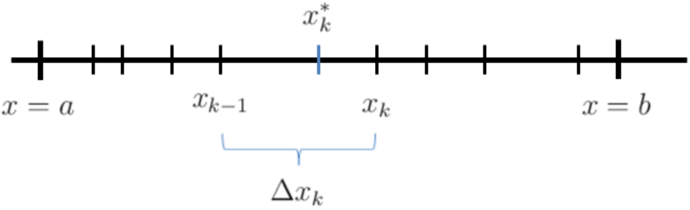
Find the **work** done by the force $F(x)$ in moving an object from $x = a$ to $x = b$.

Work Done by a **Variable Force** along the x-axis



Partition interval $[a, b]$ into N subintervals.

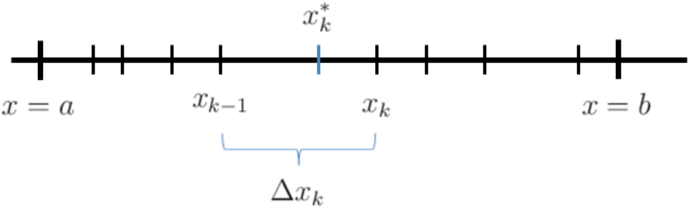
Work Done by a **Variable Force** along the x-axis



k^{th} Subinterval $[x_{k-1}, x_k]$:

Distance	$=$	(Length of Subinterval)
Force		(Assuming Δx_k is **small**)
Work	$=$	(Force) $\times$ (Distance)

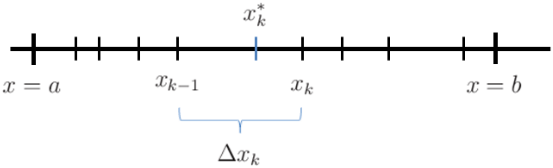
Work Done by a **Variable Force** along the x-axis



k^{th} Subinterval $[x_{k-1}, x_k]$:

Distance	$=$	(Length of Subinterval)	$=$	Δx_k
Force		(Assuming Δx_k is **small**)	$\approx$	$F(x_k^*)$
Work	$=$	(Force) $\times$ (Distance)	$\approx$	$F(x_k^*)\,\Delta x_k$

Work Done by a **Variable Force** along the x-axis

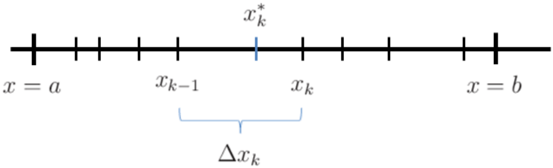


k^{th} Subinterval $[x_{k-1}, x_k]$:

Distance	$=$	(Length of Subinterval)	$=$	Δx_k
Force		(Assuming Δx_k is **small**)	$\approx$	$F(x_k^*)$
Work	$=$	(Force) $\times$ (Distance)	$\approx$	$F(x_k^*)\,\Delta x_k$

Riemann Sum: Work done by force $F(x)$ over $[a,b] \approx W_N^* := \sum_{k=1}^{N} F(x_k^*)\,\Delta x_k$

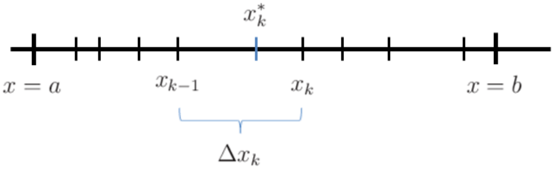
Work Done by a **Variable Force** along the x-axis



Riemann Sum: Work done by force $F(x)$ over $[a, b] \approx W_N^* := \sum_{k=1}^{N} F\left(x_k^*\right) \Delta x_k$

Integral: Work done by force $F(x)$ over $[a, b] = \lim_{N \to \infty} W_N^* = \int_a^b F(x)\, dx$

Work Done by a **Variable Force** along the x-axis

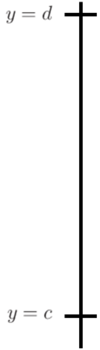


Proposition

*Let **force** $F \in C[a, b]$ s.t. $y = F(x)$. Then:*

$$\textbf{\textit{Work}} \textit{ done by force } F(x) \textit{ over interval } [a, b] = \int_a^b F(x)\, dx$$

Work Done by a **Variable Force** along the y-axis



Proposition

*Let **force** $G \in C[c, d]$ s.t. $x = G(y)$. Then:*

$$\textbf{\textit{Work}}\ \textit{done by force}\ G(y)\ \textit{over interval}\ [c, d] = \int_c^d G(y)\, dy$$

Work Done by Springs (Hooke's Law)

Proposition

*Let a spring be fixed at one end and can freely move horizontally. Moreover, the spring has **stiffness constant** $k > 0$.*

*Then the **restoring force** of the spring is:*

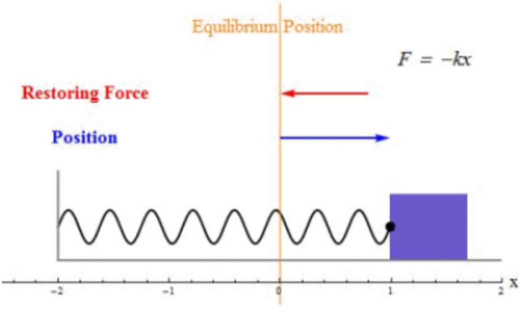
$$F(x) = -kx$$

*where $x \equiv$ distance from the spring's **natural length**.*

REMARK: For our purposes, the minus sign is not absolutely necessary.

Work Done by Springs (Demo)

(DEMO) SPRING (HOOKE'S LAW) (Click below):



PHYSICS PART II:

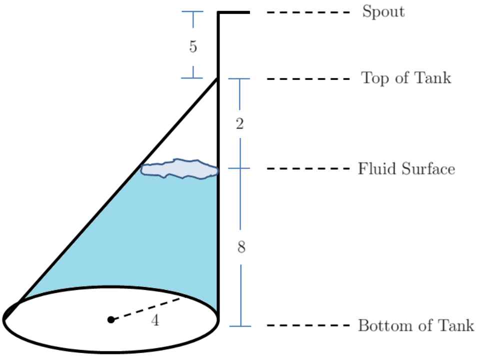
WORK PUMPING FLUIDS, FLUID FORCE

Density of Fluids

- Just saying the "**density** of a fluid" is ambiguous:
  - $\delta \equiv$ **Weight-density** of a Fluid $:=$ Weight per Volume of Fluid
    - δ is the lowercase Greek letter "delta"
    - Common units of measure: lbs/ft^3, N/m^3, dynes/cm^3
  - $\rho \equiv$ **Mass-density** of a Fluid $:=$ Mass per Volume of Fluid
    - ρ is the Greek letter "rho"
    - Common units of measure: slugs/ft^3, kg/m^3, g/cm^3
  - Relationship between Mass-Density & Weight-Density:
    - $\delta = \rho g$
    - $g \equiv$ **Acceleration of Gravity**
      Common units of measure: ft/sec^2, m/sec^2, cm/sec^2

Work Done Pumping Fluid Out of a Tank

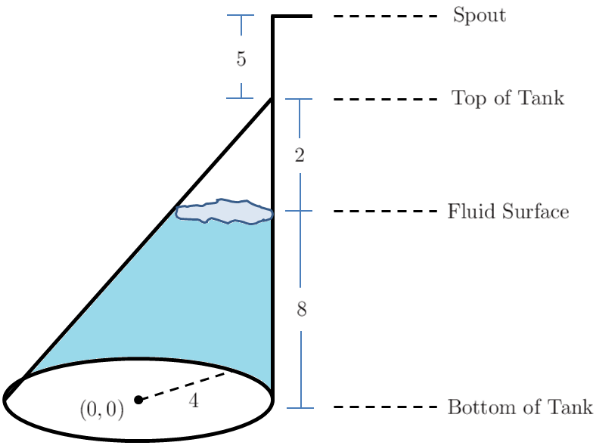
WORKED EXAMPLE:
An oblique conical tank with a spout is filled with fluid as shown below:



The **mass-density** of the fluid is denoted by ρ.
The **gravitational acceleration** is denoted by g.

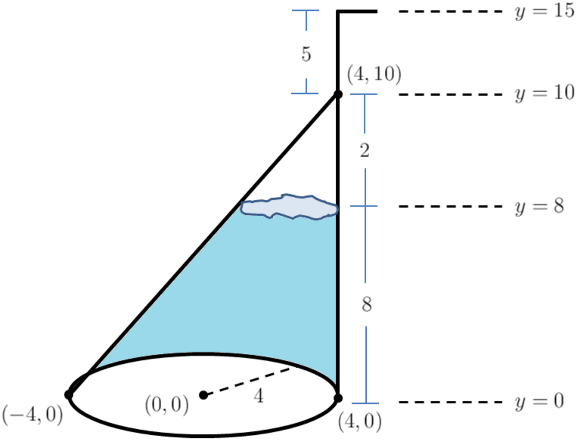
Setup integral(s) to find the **work** done pumping all the fluid out of the spout.

Work Done Pumping Fluid Out of a Tank



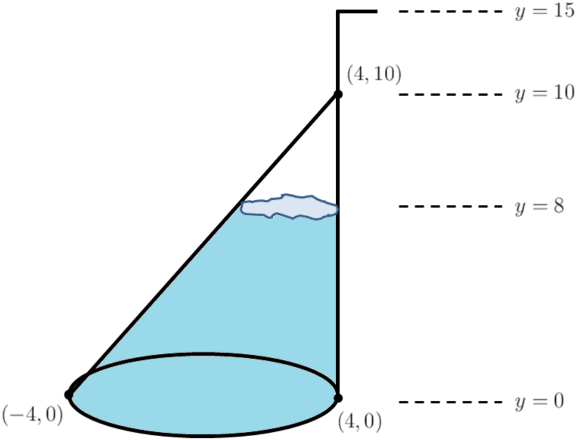
Pick a Coordinate System (by labeling one point)

Work Done Pumping Fluid Out of a Tank

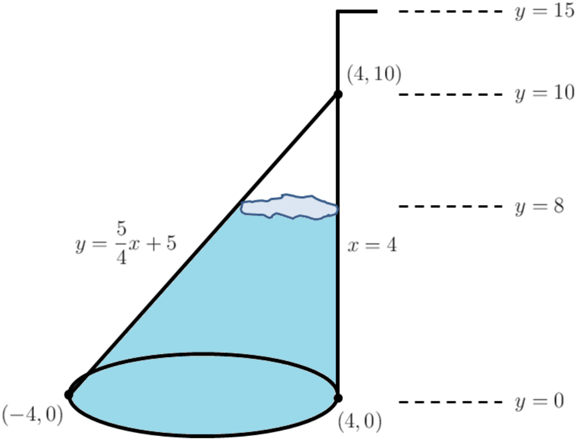


Based on the chosen point, label all other key points (especially the BP's)

Work Done Pumping Fluid Out of a Tank

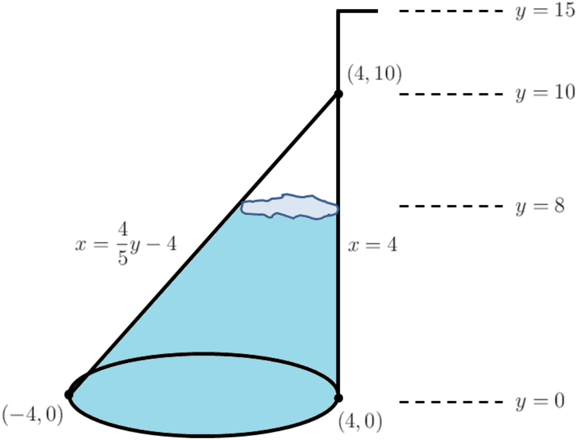


For clarity, remove some clutter.



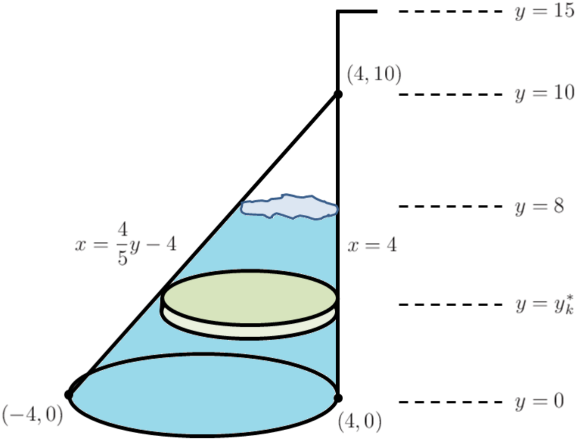
Label the Left BC's & Right BC's

Work Done Pumping Fluid Out of a Tank



Label the Left BC's & Right BC's (in terms of y)

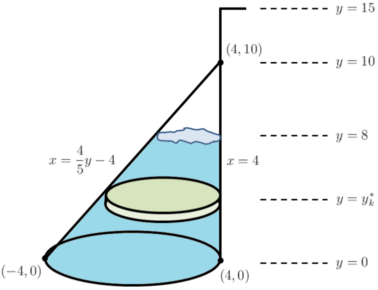
Work Done Pumping Fluid Out of a Tank



Sketch the **key element**, which is the k^{th} H-Slab of fluid.

IMPORTANT: The y-coordinate of the k^{th} H-Slab of Fluid is **always** $y = y_k^*$.

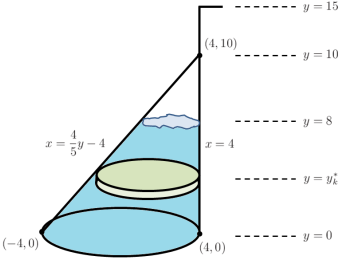
Work Done Pumping Fluid Out of a Tank



k^{th} H-Slab (of fluid):

Weight-Density	:=	(Mass-Density of Fluid) $\times$ (Gravitational Acceleration)
Thickness	:=	(Length of k^{th} subinterval)
Distance to Spout	:=	(y-coord. of Spout) $-$ (y-coord. of H-Slab)
Radius	:=	$\frac{1}{2} \times$ [(Right BC) $-$ (Left BC)]
Area	:=	$\pi \times$ (Radius)^2
Volume	:=	(Area) $\times$ (Thickness)
Weight	:=	(Weight-Density) $\times$ (Volume)
Work Done	:=	(Weight) $\times$ (Distance to Spout)

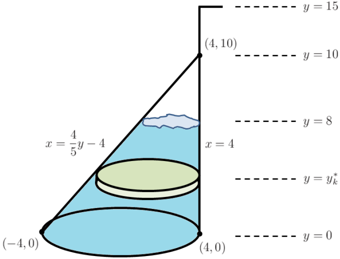
Work Done Pumping Fluid Out of a Tank



k^{th} H-Slab (of fluid):

Weight-Density	$=$	ρg
Thickness	$=$	Δy_k
Distance to Spout	$=$	$15 - y_k^*$
Radius	$=$	$\frac{1}{2}\left[4 - \left(\frac{4}{5}y_k^* - 4\right)\right]$
Area	$=$	$\frac{1}{4}\pi\left[4 - \left(\frac{4}{5}y_k^* - 4\right)\right]^2$
Volume	$=$	$\frac{1}{4}\pi\left[4 - \left(\frac{4}{5}y_k^* - 4\right)\right]^2 \Delta y_k$
Weight	$=$	$\frac{1}{4}\pi\rho g\left[4 - \left(\frac{4}{5}y_k^* - 4\right)\right]^2 \Delta y_k$
Work Done	$=$	$\frac{1}{4}\pi\rho g\left(15 - y_k^*\right)\left[4 - \left(\frac{4}{5}y_k^* - 4\right)\right]^2 \Delta y_k$

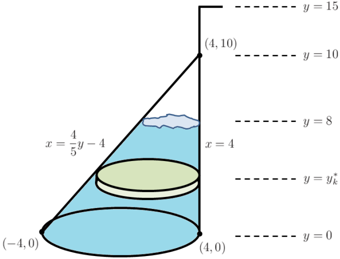
Work Done Pumping Fluid Out of a Tank



Riemann Sum: $\text{Work} \approx \sum_{k=1}^{N} \frac{1}{4}\pi\rho g\left(15 - y_k^*\right)\left[4 - \left(\frac{4}{5}y_k^* - 4\right)\right]^2 \Delta y_k$

Integral: $\text{Work} = \int_{\text{bottom } y\text{-coord. of fluid}}^{\text{top } y\text{-coord. of fluid}} \frac{1}{4}\pi\rho g(15 - y)\left[4 - \left(\frac{4}{5}y - 4\right)\right]^2\, dy$

Work Done Pumping Fluid Out of a Tank



Riemann Sum: Work $\approx \sum_{k=1}^{N} \frac{1}{4}\pi\rho g \left(15 - y_k^*\right) \left[4 - \left(\frac{4}{5}y_k^* - 4\right)\right]^2 \Delta y_k$

Integral: Work $= \boxed{\int_0^8 \frac{1}{4}\pi\rho g(15 - y) \left[4 - \left(\frac{4}{5}y - 4\right)\right]^2 \, dy}$

Work Done Pumping Fluid Out of a Tank

WeBWorK problems involving physics require **computation** of the integral:

$$
\begin{aligned}
\text{Work} &= \int_0^8 \tfrac{1}{4}\pi\rho g(15-y)\left[4-\left(\tfrac{4}{5}y-4\right)\right]^2\,dy \\
&= \pi\rho g\int_0^8 \tfrac{1}{4}(15-y)\left(8-\tfrac{4}{5}y\right)^2\,dy \\
&= \pi\rho g\int_0^8 \tfrac{1}{4}(15-y)\left(64-\tfrac{64}{5}y+\tfrac{16}{25}y^2\right)\,dy \\
&= \pi\rho g\int_0^8 (15-y)\left(16-\tfrac{16}{5}y+\tfrac{4}{25}y^2\right)\,dy \\
&= \pi\rho g\int_0^8 \left(240-48y+\tfrac{12}{5}y^2-16y+\tfrac{16}{5}y^2-\tfrac{4}{25}y^3\right)\,dy \\
&= \pi\rho g\int_0^8 \left(240-64y+\tfrac{28}{5}y^2-\tfrac{4}{25}y^3\right)\,dy \\
&= \pi\rho g\left[240y-32y^2+\tfrac{28}{15}y^3-\tfrac{1}{25}y^4\right]_{y=0}^{y=8} \\
&= \pi\rho g\left[240(8)-32(8)^2+\tfrac{28}{15}(8)^3-\tfrac{1}{25}(8)^4\right] \\
&= \pi\rho g\left[\tfrac{49792}{75}\right] \qquad \text{(Use a calculator for tedious arithmetic)} \\
&= \tfrac{49792}{75}\pi\rho g \approx 663.893333\pi\rho g \qquad \text{(For decimals, go at least 6 places)}
\end{aligned}
$$

Finally, plug in the given values for ρ and g.

Fluid Force against a Thin Plate Submerged Horizontally

Proposition

*Given a fluid with **weight-density** δ.*
*Let a **thin plate** of area A be **submerged horizontally** at depth h in the fluid.*

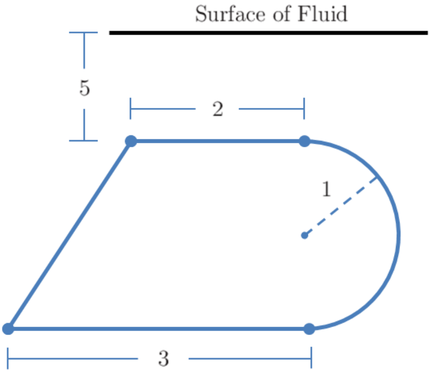
*Then, the **fluid force** against the thin plate is defined by:*

$$\Big(\textit{Fluid Force}\Big) = \Big(\textit{Pressure}\Big) \times \Big(\textit{Area}\Big)$$

$$F = (\delta h)A$$

Fluid Force against a Thin Plate Submerged **Vertically**

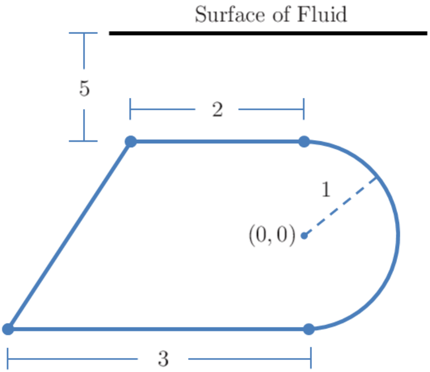
WORKED EXAMPLE: A vertical plate is submerged in fluid as shown below:



The **weight-density** of the fluid is denoted by δ.

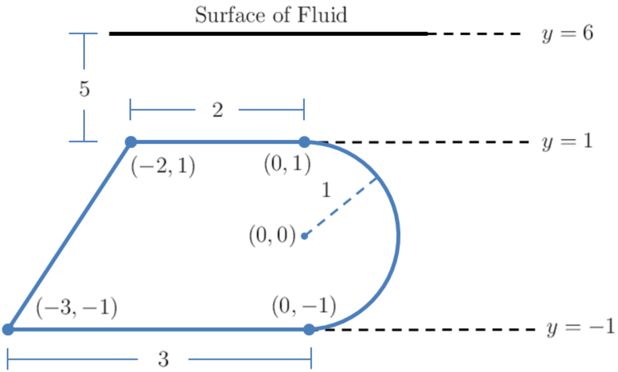
Setup integral(s) to find the **fluid force** against the vertical plate.

Fluid Force against a Thin Plate Submerged **Vertically**



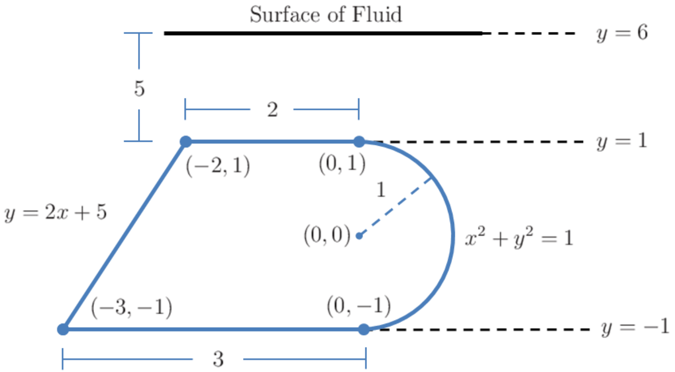
Pick a Coordinate System (by labeling one point)

Fluid Force against a Thin Plate Submerged **Vertically**



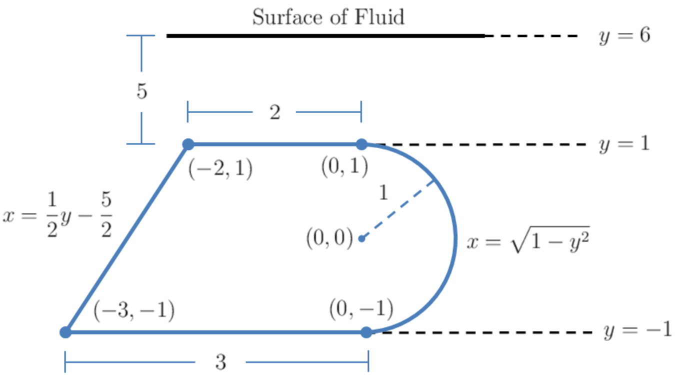
Based on the chosen point, label all other key points (especially the BP's)

Fluid Force against a Thin Plate Submerged **Vertically**



Label the Left BC's & Right BC's

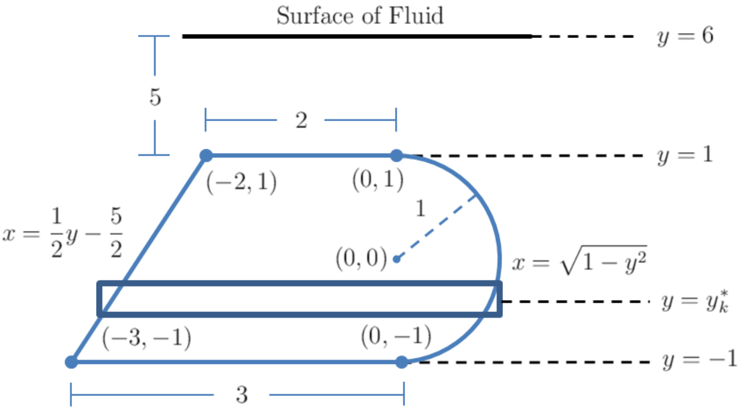
Fluid Force against a Thin Plate Submerged **Vertically**



Label the Left BC's & Right BC's (in terms of y)

For the semicircle, the **positive** root was chosen since $x \geq 0$

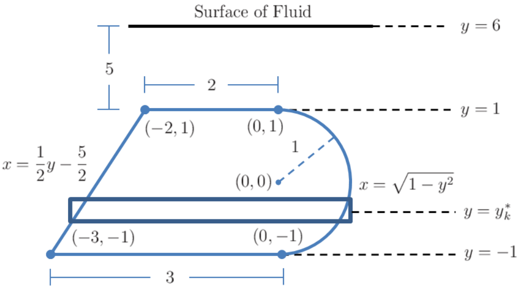
Fluid Force against a Thin Plate Submerged **Vertically**



Sketch the **key element**, which is the k^{th} H-Rect of the plate.

IMPORTANT: The y-coordinate of the k^{th} H-Rect is **always** $y = y_k^*$.

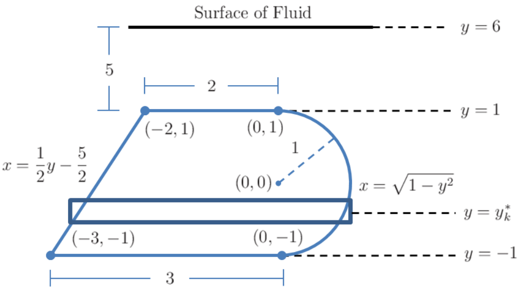
Fluid Force against a Thin Plate Submerged **Vertically**



k^{th} H-Rect (of plate):

Weight-Density	$:=$	(Weight per Volume of Fluid)
Width	$:=$	(Length of k^{th} subinterval)
Depth	$:=$	(y-coord. of Surface) $-$ (y-coord. of H-Rect)
Length	$:=$	(Right BC of Plate) $-$ (Left BC of Plate)
Pressure	$:=$	(Weight-Density) $\times$ (Depth)
Area	$:=$	(Length) $\times$ (Width)
Fluid Force	$:=$	(Pressure) $\times$ (Area)

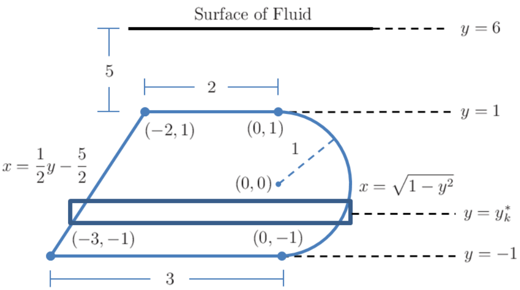
Fluid Force against a Thin Plate Submerged **Vertically**



k^{th} H-Rect (of plate):

Weight-Density	$=$	δ
Width	$=$	Δy_k
Depth	$=$	$6 - y_k^*$
Length	$=$	$\sqrt{1 - (y_k^*)^2} - \left(\frac{1}{2}y_k^* - \frac{5}{2}\right)$
Pressure	$=$	$\delta(6 - y_k^*)$
Area	$=$	$\left[\sqrt{1 - (y_k^*)^2} - \left(\frac{1}{2}y_k^* - \frac{5}{2}\right)\right] \Delta y_k$
Fluid Force	$=$	$\delta(6 - y_k^*) \left[\sqrt{1 - (y_k^*)^2} - \left(\frac{1}{2}y_k^* - \frac{5}{2}\right)\right] \Delta y_k$

Fluid Force against a Thin Plate Submerged **Vertically**



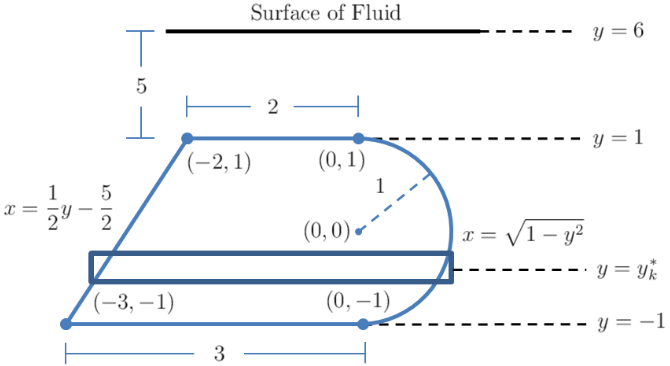
Riemann Sum:

$$\text{FluidForce(Plate)} \approx \sum_{k=1}^{N} \delta(6 - y_k^*) \left[\sqrt{1 - (y_k^*)^2} - \left(\frac{1}{2} y_k^* - \frac{5}{2} \right) \right] \Delta y_k$$

Integral:

$$\text{FluidForce(Plate)} = \int_{\text{bottom } y\text{-coord. of plate}}^{\text{top } y\text{-coord. of plate}} \delta(6 - y) \left[\sqrt{1 - y^2} - \left(\frac{1}{2} y - \frac{5}{2} \right) \right] \, dy$$

Fluid Force against a Thin Plate Submerged **Vertically**



$$\text{FluidForce(Plate)} = \boxed{\int_{-1}^{1} \delta(6-y)\left[\sqrt{1-y^2} - \left(\frac{1}{2}y - \frac{5}{2}\right)\right]\, dy}$$

Fluid Force against a Thin Plate Submerged **Vertically**

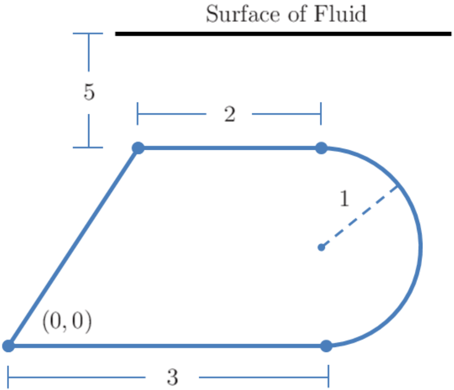
QUESTION: What if a different coordinate system is chosen???

Fluid Force against a Thin Plate Submerged **Vertically**

QUESTION: What if a different coordinate system is chosen???

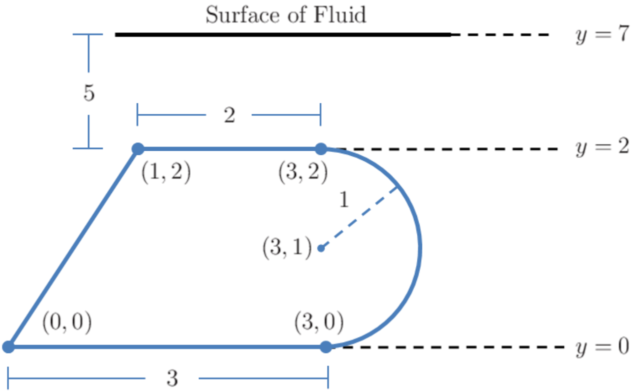
ANSWER: The integral expression will change, but the value will be same!

Fluid Force against a Thin Plate Submerged **Vertically**



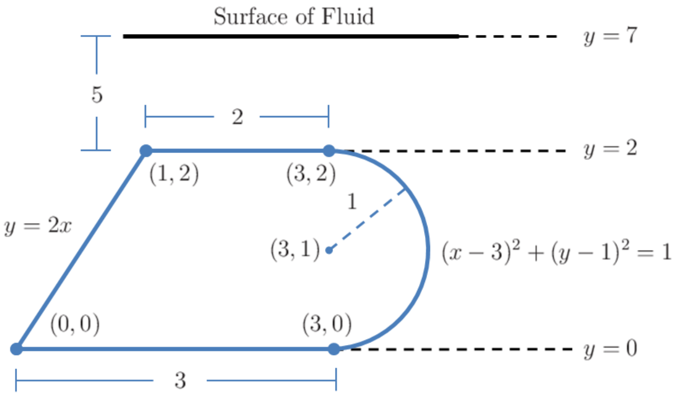
Pick a different Coordinate System.
Now, the point $(0, 0)$ is at the bottom-left BP of thin plate.

Fluid Force against a Thin Plate Submerged **Vertically**



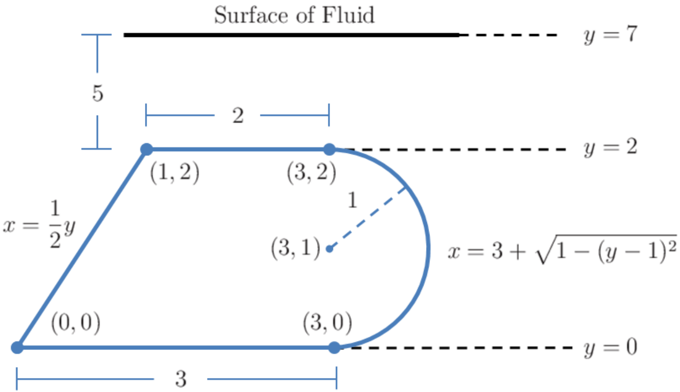
Based on the chosen point, label all other key points (especially the BP's)

Fluid Force against a Thin Plate Submerged **Vertically**



Label the Left BC's & Right BC's

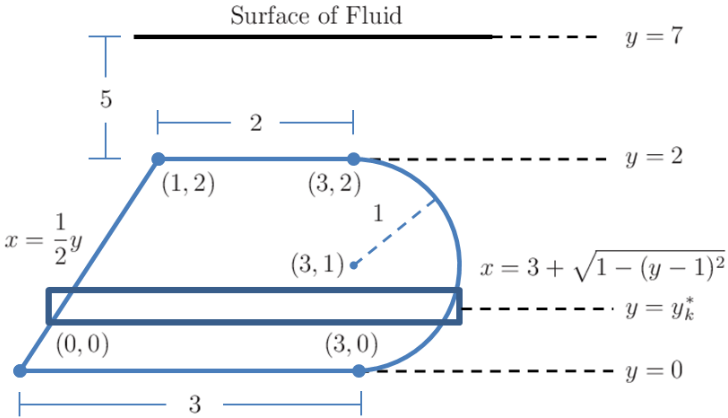
Fluid Force against a Thin Plate Submerged **Vertically**



Label the Left BC's & Right BC's (in terms of y)

For the semicircle, the **positive** root was chosen since $x \geq 3$

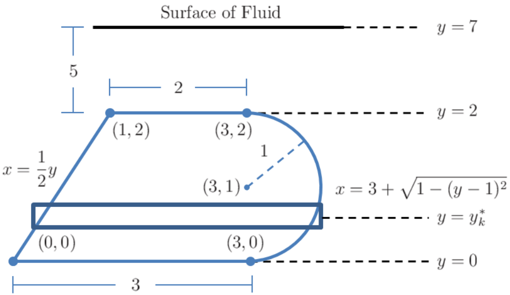
Fluid Force against a Thin Plate Submerged **Vertically**



Sketch the **key element**, which is the k^{th} H-Rect of the plate.

IMPORTANT: The y-coordinate of the k^{th} H-Rect is **always** $y = y_k^*$.

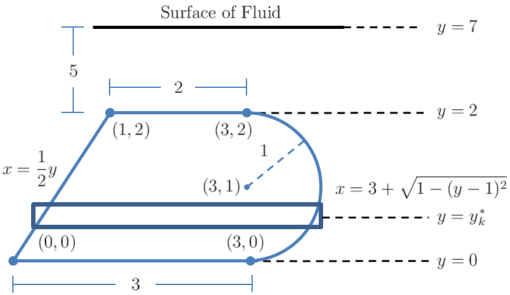
Fluid Force against a Thin Plate Submerged **Vertically**



k^{th} H-Rect (of plate):

Weight-Density	$:=$	(Weight per Volume of Fluid)
Width	$:=$	(Length of k^{th} subinterval)
Depth	$:=$	(y-coord. of Surface) $-$ (y-coord. of H-Rect)
Length	$:=$	(Right BC of Plate) $-$ (Left BC of Plate)
Pressure	$:=$	(Weight-Density) $\times$ (Depth)
Area	$:=$	(Length) $\times$ (Width)
Fluid Force	$:=$	(Pressure) $\times$ (Area)

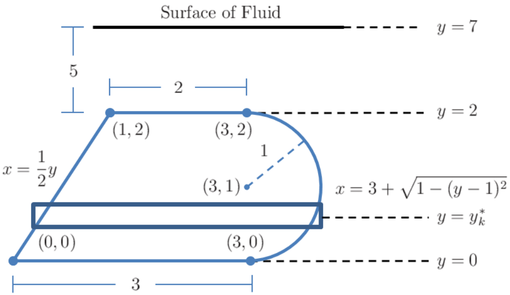
Fluid Force against a Thin Plate Submerged **Vertically**



k^{th} H-Rect (of plate):

Weight-Density	$=$	δ
Width	$=$	Δy_k
Depth	$=$	$7 - y_k^*$
Length	$=$	$3 + \sqrt{1 - (y_k^* - 1)^2} - \frac{1}{2}y_k^*$
Pressure	$=$	$\delta(7 - y_k^*)$
Area	$=$	$\left[3 + \sqrt{1 - (y_k^* - 1)^2} - \frac{1}{2}y_k^*\right] \Delta y_k$
Fluid Force	$=$	$\delta(7 - y_k^*) \left[3 + \sqrt{1 - (y_k^* - 1)^2} - \frac{1}{2}y_k^*\right] \Delta y_k$

Fluid Force against a Thin Plate Submerged **Vertically**



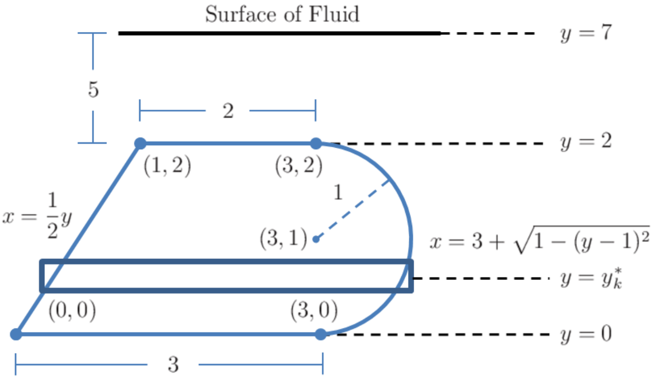
Riemann Sum:

$$\text{FluidForce(Plate)} \approx \sum_{k=1}^{N} \delta(7 - y_k^*) \left[3 + \sqrt{1 - (y_k^* - 1)^2} - \frac{1}{2}y_k^*\right] \Delta y_k$$

Integral:

$$\text{FluidForce(Plate)} = \int_{\text{bottom } y\text{-coord. of plate}}^{\text{top } y\text{-coord. of plate}} \delta(7 - y) \left[3 + \sqrt{1 - (y - 1)^2} - \frac{1}{2}y\right] dy$$

Fluid Force against a Thin Plate Submerged **Vertically**



$$\text{FluidForce(Plate)} = \boxed{\int_0^2 \delta(7-y)\left[3+\sqrt{1-(y-1)^2}-\frac{1}{2}y\right]\,dy}$$

Fluid Force against a Thin Plate Submerged **Vertically**

So, even though two different coordinate systems were used (which resulted in two different integral expressions), the values of both integral expressions are the same:

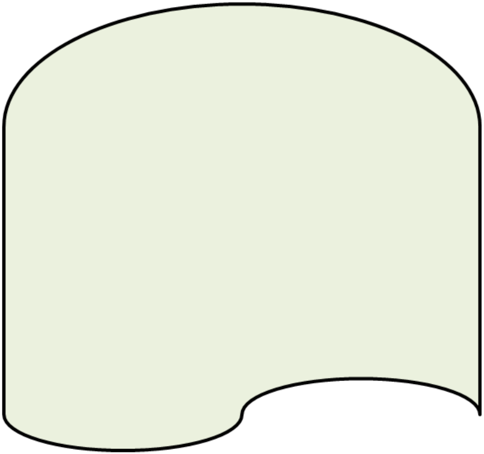
$$\int_{-1}^{1} \delta(6-y)\left[\sqrt{1-y^2}-\left(\frac{1}{2}y-\frac{5}{2}\right)\right]\,dy = \left(3\pi+\frac{91}{3}\right)\delta$$

$$\int_{0}^{2} \delta(7-y)\left[3+\sqrt{1-(y-1)^2}-\frac{1}{2}y\right]\,dy = \left(3\pi+\frac{91}{3}\right)\delta$$

PHYSICS PART III:

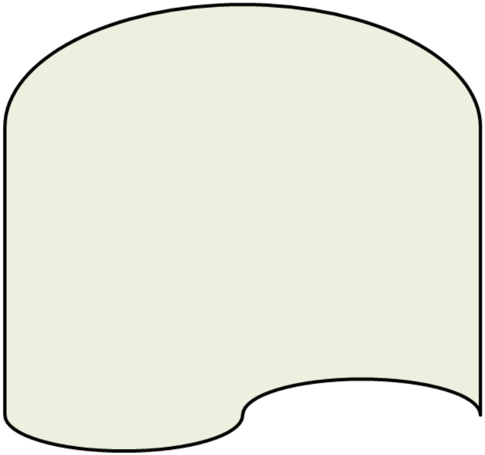
CENTROIDS

Centroids



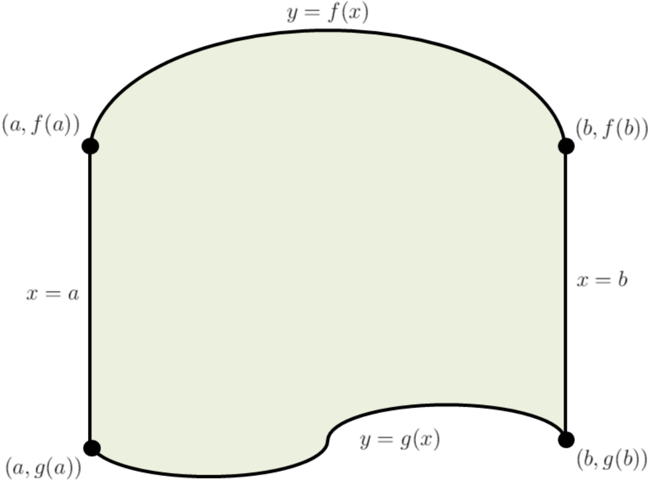
Given **lamina** given above and uniform (constant) **mass-density** ρ.

Centroids



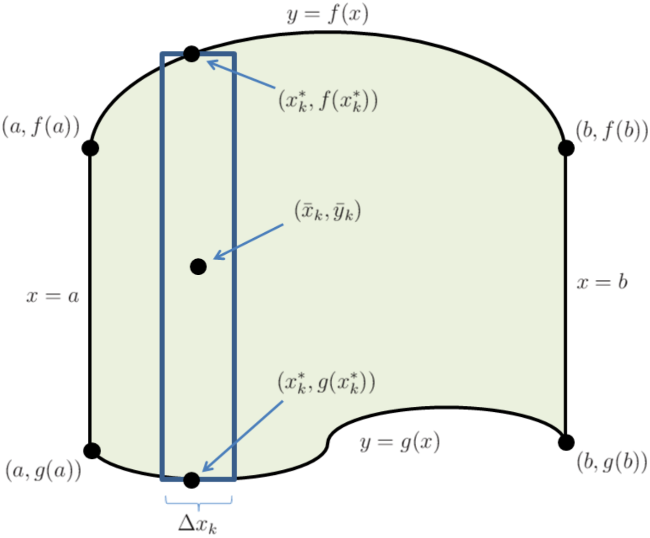
Setup integrals to find the **moments** about the x-axis & y-axis of lamina R.

Centroids



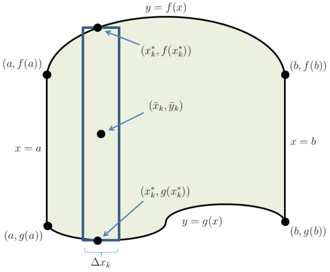
As usual, label all BP's and BC's.

Centroids



Key element: V-Rect

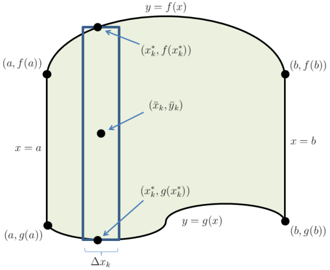
Centroids



k^{th} V-Rect:

Mass-Density	:=	(Mass per Area of Lamina)
Width	:=	(Length of k^{th} subinterval)
Height	:=	(Top BC) − (Bottom BC)
Centroid $(\bar{x}_k, \bar{y}_k)$	:=	(Geometric Center of V-Rect)
Mass Δm_k	:=	(Mass-Density) $\times$ (Area)
Moment about y-axis	:=	$\bar{x}_k \Delta m_k$
Moment about x-axis	:=	$\bar{y}_k \Delta m_k$

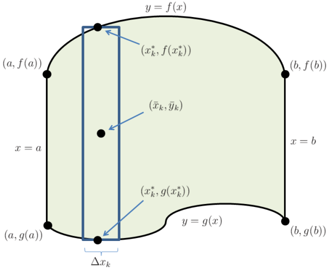
Centroids



k^{th} V-Rect:

Mass-Density	$=$	ρ
Width	$=$	Δx_k
Height	$=$	$f(x_k^*) - g(x_k^*)$
Centroid $(\bar{x}_k, \bar{y}_k)$	$=$	$\left(x_k^*, \frac{1}{2}\left[f(x_k^*) + g(x_k^*)\right]\right)$
Mass Δm_k	$=$	$\rho[f(x_k^*) - g(x_k^*)]\Delta x_k$
Moment about y-axis	$=$	$\rho x_k^*[f(x_k^*) - g(x_k^*)]\Delta x_k$
Moment about x-axis	$=$	$\frac{1}{2}\rho[f(x_k^*) + g(x_k^*)][f(x_k^*) - g(x_k^*)]\Delta x_k$

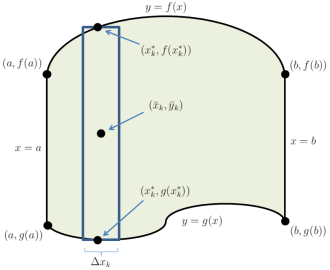
Centroids



k^{th} V-Rect:

Mass-Density	$=$	ρ
Width	$=$	Δx_k
Height	$=$	$f(x_k^*) - g(x_k^*)$
Centroid $(\bar{x}_k, \bar{y}_k)$	$=$	$\left(x_k^*, \frac{1}{2}\left[f(x_k^*) + g(x_k^*)\right]\right)$
Mass Δm_k	$=$	$\rho[f(x_k^*) - g(x_k^*)]\Delta x_k$
Moment about y-axis	$=$	$\rho x_k^*[f(x_k^*) - g(x_k^*)]\Delta x_k$
Moment about x-axis	$=$	$\frac{1}{2}\rho\left([f(x_k^*)]^2 - [g(x_k^*)]^2\right)\Delta x_k$

Centroids

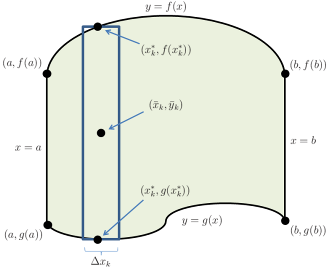


Mass of Lamina $\approx \sum_{k=1}^{N} \rho[f(x_k^*) - g(x_k^*)]\Delta x_k$

Moment about y-axis $\approx \sum_{k=1}^{N} \rho x_k^*[f(x_k^*) - g(x_k^*)]\Delta x_k$

Moment about x-axis $\approx \sum_{k=1}^{N} \frac{1}{2}\rho \left([f(x_k^*)]^2 - [g(x_k^*)]^2\right) \Delta x_k$

Centroids



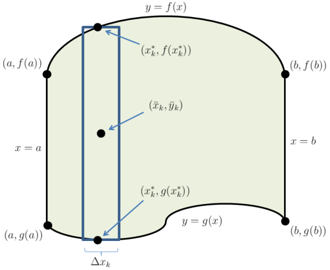
$$\text{Mass of Lamina} \equiv m = \int_{\text{smallest } x\text{-coord.}}^{\text{largest } x\text{-coord.}} \rho[f(x) - g(x)]\, dx$$

$$\text{Moment about } y\text{-axis} \equiv M_y = \int_{\text{smallest } x\text{-coord.}}^{\text{largest } x\text{-coord.}} \rho x[f(x) - g(x)]\, dx$$

$$\text{Moment about } x\text{-axis} \equiv M_x = \int_{\text{smallest } x\text{-coord.}}^{\text{largest } x\text{-coord.}} \frac{1}{2}\rho\left([f(x)]^2 - [g(x)]^2\right)\, dx$$

$$\text{Centroid of Lamina} \equiv (\bar{x}, \bar{y}) = \left(\frac{M_y}{m}, \frac{M_x}{m}\right)$$

Centroids



Mass of Lamina $\equiv m = \displaystyle\int_a^b \rho[f(x) - g(x)]\,dx$

Moment about y-axis $\equiv M_y = \displaystyle\int_a^b \rho x[f(x) - g(x)]\,dx$

Moment about x-axis $\equiv M_x = \displaystyle\int_a^b \frac{1}{2}\rho\left([f(x)]^2 - [g(x)]^2\right)\,dx$

Centroid of Lamina $\equiv (\bar{x}, \bar{y}) = \left(\dfrac{M_y}{m}, \dfrac{M_x}{m}\right)$

Fin.