Integration by Parts (IBP)

Calculus II

Josh Engwer

TTU

24 February 2014

Integration by Parts (Motivation)

Recall from Calculus I that there's a sum/difference rule for integration:

$$\int \Big[f(x) \pm g(x)\Big]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$$

However, there is no ”product rule” for integration:

i.e. $\displaystyle \int f(x)g(x)\, dx \neq \left(\int f(x)\, dx\right)\left(\int g(x)\, dx\right)$ in general

Integration by Parts (Motivation)

Recall from Calculus I that there's no "product rule" for integration:

$$\text{i.e.} \int f(x)g(x)\,dx \neq \left(\int f(x)\,dx\right)\left(\int g(x)\,dx\right) \text{ in general}$$

Integration by Parts (IBP) is the next best thing to a "product rule".

Integration by Parts (Derivation)

Recall the product rule for **derivatives**:

$$\frac{d}{dx}[uv] = v\frac{du}{dx} + u\frac{dv}{dx}$$

(where u, v are differentiable functions of x)

$$\implies d[uv] = v\,du + u\,dv$$

$$\implies \int d[uv] = \int v\,du + \int u\,dv$$

$$\implies uv = \int v\,du + \int u\,dv$$

$$\implies \boxed{\int u\,dv = uv - \int v\,du}$$

Integration by Parts (Definition)

Proposition

Let $u, v \in C^1[a,b]$ *s.t.* $u(x), v(x)$ *are not piecewise functions. Then:*

$$\int u\ dv = uv - \int v\ du \qquad \textit{(Indefinite Integral Form)}$$

$$\int_a^b u\ dv = \Big[uv\Big]_{x=a}^{x=b} - \int_a^b v\ du \qquad \textit{(Definite Integral Form)}$$

Integration by Parts (Purpose)

$$\text{IBP:} \quad \underbrace{\int u\,dv}_{\text{Hard}} = uv - \underbrace{\int v\,du}_{\text{Easier}}$$

But the question is: How to choose function u & differential dv???

Choosing the right function u (LIPTE Heuristic)

LIPTE Heuristic:

Whichever function type comes first in the following list choose as u:

Letter	Function Type	Example Functions
L	Logarithms	$\ln x, \log y, \log_8 t, \ldots$
I	Inverse Trig	$\arcsin x, \arctan y, \text{arcsec}\ t, \ldots$
P	Polynomials	$x, y^2, 5t^3, \ldots$
T	Trig Functions	$\sin x, \tan\theta, \sec\omega, \ldots$
E	Exponentials	$e^x, 2^y, \left(\frac{1}{3}\right)^t, (-4)^x, \ldots$

There's no **R** in LIPTE, so **rational fcns** & **roots** are disqualified to be u.

Once u is chosen, the remainder of the integrand must be dv.

If integrand involves **only rational fcns and/or roots**, IBP is no good:

$$\int \frac{dx}{(x-1)(x+7)}, \quad \int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}, \quad \int \frac{1+\sqrt{x}}{x^2+1}\,dx, \quad \text{etc...}$$

IBP involving Compositions

Definition

A function f is a **light composition** $\iff$ f has form $f(ax+b)$ s.t. $a, b \in \mathbb{R}$.

EXAMPLES: $\sin(\pi\theta - 4), \dfrac{1}{3t-2}, \sqrt{2z-1}, \arctan(x\sqrt{5}), e^{7y+3}, (9w-8)^4, \ldots$

Definition

A function f is a **heavy composition** $\iff$ f is NOT a light composition.

EXAMPLES: $\sin(2\arccos x), \dfrac{1}{\ln x}, \sqrt{e^y}, \arctan(1/x), e^{\sqrt{t}}, \ln(\ln w), \ldots$

- For **light compositions**, immediately perform IBP, afterwards u-substitution may be necessary.
- For **heavy compositions**, consider u-substitution first, afterwards perform IBP.

IBP (Example)

WORKED EXAMPLE: Evaluate $I = \int x \cos x \, dx$.

Observe that the integrand involves a polynomial & trig fcn.

Hence, by the LIPTE Heuristic, pick $u = x$.

IBP: Let $\begin{cases} u = x \\ dv = \cos x \, dx \end{cases} \implies \begin{cases} du = dx \\ v = \int \cos x \, dx = \sin x \end{cases}$

$$\therefore I = \int u \, dv \overset{IBP}{=} uv - \int v \, du = x \sin x - \int \sin x \, dx = \boxed{x \sin x + \cos x + C}$$

IBP (Example)

WORKED EXAMPLE: Evaluate $I = \int_0^{\pi/2} x \cos x \, dx$.

Observe that the integrand involves a polynomial & trig fcn.

Hence, by the LIPTE Heuristic, pick $u = x$.

IBP: Let $\begin{cases} u = x \\ dv = \cos x \, dx \end{cases} \implies \begin{cases} du = dx \\ v = \int \cos x \, dx = \sin x \end{cases}$

$$
\begin{aligned}
I &= \int_0^{\pi/2} u \, dv \overset{IBP}{=} \Big[uv\Big]_{x=0}^{x=\pi/2} - \int_0^{\pi/2} v \, du \\
&= \Big[x \sin x\Big]_{x=0}^{x=\pi/2} - \int_0^{\pi/2} \sin x \, dx \\
&= (\pi/2)\sin(\pi/2) - (0)\sin(0) - \Big[-\cos x\Big]_{x=0}^{x=\pi/2} \\
&= \pi/2 - \Big[-\cos(\pi/2) - (-\cos(0))\Big] \\
&= \pi/2 - \Big[-(0) + 1\Big] \\
&= \boxed{\pi/2 - 1}
\end{aligned}
$$

Tabular Integration (with an Indefinite Integral)

If u is a polynomial, **tabular integration** may be more efficient.
Tabular integration is the repeated use of IBP, but in the form of a table:

WORKED EXAMPLE: Evaluate $I = \int x \cos x \, dx$ using tabular integration.

Via LIPTE Heuristic, let $u = x$ and $dv = \cos x \, dx$.

$+$		u		dv		Terms to Sum
	$\searrow$					
$-$		x		$\cos x$		
	$\searrow$		$\searrow$			
$+$		$\frac{d}{dx}[x] = 1$		$\int \cos x \, dx = \sin x$	$\longrightarrow$	$+x \sin x = x \sin x$
	$\searrow$		$\searrow$			
$-$		$\frac{d}{dx}[1] = 0$		$\int \sin x \, dx = -\cos x$	$\longrightarrow$	$-(1)(-\cos x) = \cos x$
	$\searrow$		$\searrow$			
$+$		$\frac{d}{dx}[0] = 0$		$\int (-\cos x) \, dx = -\sin x$	$\longrightarrow$	$+(0)(-\sin x) = 0$

$$I \overset{TAB}{=} \boxed{x \sin x + \cos x + C}$$

Tabular Integration (with a Definite Integral)

If u is a polynomial, **tabular integration** may be more efficient.
Tabular integration is the repeated use of IBP, but in the form of a table:

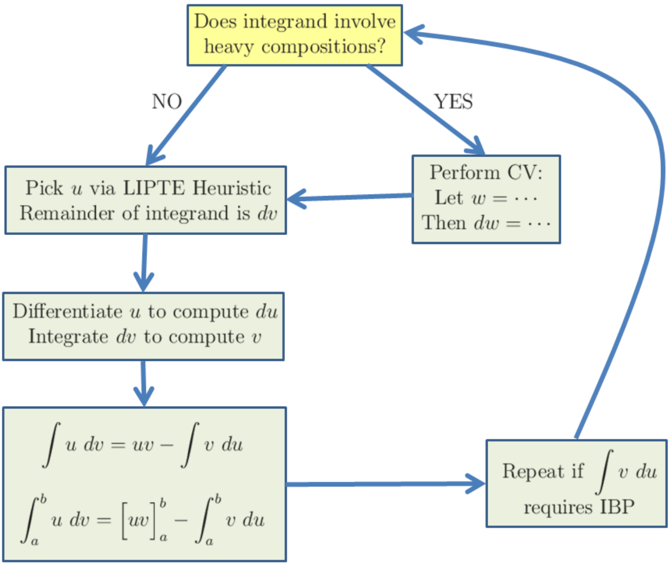
WORKED EXAMPLE: Evaluate $I = \int_0^{\pi/2} x \cos x \, dx$ using tabular integration.

Via LIPTE Heuristic, let $u = x$ and $dv = \cos x \, dx$.

$+$		u		dv		Terms to Sum
	$\searrow$					
$-$		x		$\cos x$		
	$\searrow$		$\searrow$			
$+$		1		$\sin x$	$\longrightarrow$	$+x \sin x = x \sin x$
	$\searrow$		$\searrow$			
$-$		0		$-\cos x$	$\longrightarrow$	$-(1)(-\cos x) = \cos x$
	$\searrow$		$\searrow$			
$+$		0		$-\sin x$	$\longrightarrow$	$+(0)(-\sin x) = 0$

$$I \overset{TAB}{=} \Big[x \sin x + \cos x \Big]_{x=0}^{x=\pi/2} \overset{FTC}{=} \left(\tfrac{\pi}{2}\right) \sin\left(\tfrac{\pi}{2}\right) + \cos\left(\tfrac{\pi}{2}\right) - (0)\sin(0) - \cos(0) = \boxed{\frac{\pi}{2} - 1}$$

IBP (Flowchart)



Fin

Fin.