Trig Integrals, Trig Substitution

Calculus II

Josh Engwer

TTU

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PART I:

TRIG INTEGRALS

Pascal's Triangle

Recall from Algebra that **Pascal's Triangle** is useful in expanding $(x+y)^n$:

$$\begin{array}{ccccccccccccc}
 & & & & & 1 & & & & & & & \leftarrow n=0 \\
 & & & & 1 & & 1 & & & & & & \leftarrow n=1 \\
 & & & 1 & & 2 & & 1 & & & & & \leftarrow n=2 \\
 & & 1 & & 3 & & 3 & & 1 & & & & \leftarrow n=3 \\
 & 1 & & 4 & & 6 & & 4 & & 1 & & & \leftarrow n=4 \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \leftarrow n=5 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Binomial Theorem: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^n$

where $\binom{n}{k}$ is the k^{th} entry in the n^{th} row of Pascal's Triangle.

Examples: $\binom{2}{0} = 1, \quad \binom{3}{3} = 1, \quad \binom{5}{2} = 10$

Pascal's Triangle

WORKED EXAMPLE: Expand $(x+y)^5$.

					1					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1
⋮		⋮		⋮		⋮		⋮		⋮

$$(x+y)^5 = 1x^5y^0 + 5x^4y^1 + 10x^3y^2 + 10x^2y^3 + 5x^1y^4 + 1x^0y^5$$

$$\therefore\ (x+y)^5 = \boxed{x^5 + 5x^4y + 10x^3y^2 + 10x^2y^3 + 5xy^4 + y^5}$$

Much faster than multiplying $(x+y)(x+y)(x+y)(x+y)(x+y)$

Pascal's Triangle

WORKED EXAMPLE: Expand $(x^2+4y)^3$.

					1					
				1		1				
			1		2		1			
		1		3		3		1		
	1		4		6		4		1	
1		5		10		10		5		1
⋮		⋮		⋮		⋮		⋮		⋮

$$(x^2+4y)^3 = 1\,(x^2)^3\,(4y)^0 + 3\,(x^2)^2\,(4y)^1 + 3\,(x^2)^1\,(4y)^2 + 1\,(x^2)^0\,(4y)^3$$

$$\therefore\ (x^2+4y)^3 = \boxed{x^6 + 12x^4y + 48x^2y^2 + 64y^3}$$

Much faster than multiplying $(x^2+4y)\,(x^2+4y)\,(x^2+4y)$

Trig Integrals involving Sine & Cosine (Toolkit)

TASK: Evaluate $I = \displaystyle\int \sin^m x \cos^n x\, dx$, where $m, n \in \bar{\mathbb{N}} := \{0, 1, 2, 3, \cdots\}$

- Binomial Theorem: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^n$
- Pascal's Triangle
- Exponent Rules: $x^m x^n = x^{m+n}$, $(x^m)^n = x^{mn}$
- Pythagorean Identity: $\sin^2 x + \cos^2 x = 1$
- Half-Angle Identities: $\cos^2 x = \frac{1}{2}\left[1 + \cos(2x)\right]$, $\sin^2 x = \frac{1}{2}\left[1 - \cos(2x)\right]$
- Negative-Angle Identities: $\cos(-x) = \cos x$, $\sin(-x) = -\sin x$
- Change of Variables (CV)
- $\int u^n\, du = \frac{1}{n+1} u^{n+1} + C$

Trig Integrals involving Sine & Cosine (CASE I)

TASK: Evaluate $I = \int \sin^m x \cos^n x \, dx$, where $m, n \in \bar{\mathbb{N}} := \{0, 1, 2, 3, \cdots\}$

CASE I: m is odd $\implies m = 2k + 1$, where $k \in \bar{\mathbb{N}}$

$$\begin{aligned} I &= \int \sin^{2k+1} x \cos^n x \, dx \\ &= \int \sin^{2k} x \cos^n x \sin x \, dx \\ &= \int \left(\sin^2 x\right)^k \cos^n x \sin x \, dx \\ &= \int \left(1 - \cos^2 x\right)^k \cos^n x \sin x \, dx \\ &\overset{CV}{=} \int \left(1 - u^2\right)^k u^n (-du) \\ &= \text{(Expand, then Distribute, then Integrate)} \end{aligned}$$

CV: Let $u = \cos x \implies du = -\sin x \, dx \implies \sin x \, dx = -du$

Trig Integrals involving Sine & Cosine (CASE II)

TASK: Evaluate $I = \displaystyle\int \sin^m x \cos^n x \, dx$, where $m, n \in \bar{\mathbb{N}} := \{0, 1, 2, 3, \cdots\}$

CASE II: n is odd $\implies n = 2k + 1$, where $k \in \bar{\mathbb{N}}$

$$
\begin{aligned}
I &= \int \sin^m x \cos^{2k+1} x \, dx \\
&= \int \sin^m x \cos^{2k} x \cos x \, dx \\
&= \int \sin^m x \left(\cos^2 x\right)^k \cos x \, dx \\
&= \int \sin^m x \left(1 - \sin^2 x\right)^k \cos x \, dx \\
&\overset{CV}{=} \int u^m \left(1 - u^2\right)^k \, du \\
&= \text{(Expand, then Distribute, then Integrate)}
\end{aligned}
$$

CV: Let $u = \sin x \implies du = \cos x \, dx \implies \cos x \, dx = du$

Trig Integrals involving Sine & Cosine (CASE III)

TASK: Evaluate $I = \int \sin^m x \cos^n x \, dx$, where $m, n \in \bar{\mathbb{N}} := \{0, 1, 2, 3, \cdots\}$

CASE III: m is even AND $n = 0 \implies m = 2k$, where $k \in \mathbb{N} := \{1, 2, 3, \cdots\}$

$$
\begin{aligned}
I &= \int \sin^{2k} x \, dx \\
&= \int \left(\sin^2 x\right)^k \, dx \\
&= \int \left[\frac{1}{2}\left(1 - \cos(2x)\right)\right]^k \, dx \\
&= \text{(Expand \& Use Half-Angle Identity as needed, then Integrate)}
\end{aligned}
$$

Trig Integrals involving Sine & Cosine (CASE IV)

TASK: Evaluate $I = \int \sin^m x \cos^n x \, dx$, where $m, n \in \bar{\mathbb{N}} := \{0, 1, 2, 3, \cdots\}$

CASE IV: $m = 0$ AND n is even $\implies n = 2k$, where $k \in \mathbb{N} := \{1, 2, 3, \cdots\}$

$$
\begin{aligned}
I &= \int \cos^{2k} x \, dx \\
&= \int \left(\cos^2 x\right)^k \, dx \\
&= \int \left[\frac{1}{2}\left(1 + \cos(2x)\right)\right]^k \, dx \\
&= \text{(Expand \& Use Half-Angle Identity as needed, then Integrate)}
\end{aligned}
$$

Trig Integrals involving Sine & Cosine (CASE X)

TASK: Evaluate $I = \int \sin^m x \cos^n x \, dx$, where $m, n \in \bar{\mathbb{N}} := \{0, 1, 2, 3, \cdots\}$

CASE X: m, n are both even such that $m \geq 4$ and $n \geq 4$ (examples below)

$$\int \sin^4 x \cos^4 x \, dx, \int \sin^4 x \cos^6 x \, dx, \int \sin^6 x \cos^4 x \, dx, \int \sin^6 x \cos^6 x \, dx,$$
$$\int \sin^4 x \cos^8 x \, dx, \int \sin^8 x \cos^6 x \, dx, \int \sin^{12} x \cos^4 x \, dx, \int \sin^{50} x \cos^{100} x \, dx, \cdots$$

- Such integrals require the use of **Product-to-Sum Identities**, creating extreme tedium:
  - $\sin A \cos B = \frac{1}{2} [\sin(A + B) + \sin(A - B)]$
  - $\cos A \cos B = \frac{1}{2} [\cos(A + B) + \cos(A - B)]$
  - $\sin A \sin B = -\frac{1}{2} [\cos(A + B) - \cos(A - B)]$
- Therefore, this case will NOT be considered in this course.

Trig Integrals involving Tangent & Secant (Toolkit)

TASK: Evaluate $I = \displaystyle\int \tan^m x \sec^n x \, dx,$ where $m, n \in \bar{\mathbb{N}} := \{0, 1, 2, 3, \cdots\}$

- Binomial Theorem: $(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^n$
- Pascal's Triangle
- Exponent Rules: $x^m x^n = x^{m+n}$, $(x^m)^n = x^{mn}$
- Pythagorean Identity: $1 + \tan^2 x = \sec^2 x$
- Quotient Identity: $\tan x = \frac{\sin x}{\cos x}$
- Negative-Angle Identities: $\tan(-x) = -\tan x$, $\sec(-x) = \sec x$
- Change of Variables (CV)
- Integration by Parts (IBP)
- $\int u^n \, du = \frac{1}{n+1} u^{n+1} + C$
- $\int \tan x \, dx = -\ln|\cos x| + C$
- $\int \sec x \, dx = \ln|\sec x + \tan x| + C$

Trig Integrals involving Tangent & Secant (CASE I)

TASK: Evaluate $I = \displaystyle\int \tan^m x \sec^n x \, dx$, where $m, n \in \bar{\mathbb{N}} := \{0, 1, 2, 3, \cdots\}$

CASE I: n is even $\implies n = 2k$, where $k \in \mathbb{N}$

$$\begin{aligned}
I &= \int \tan^m x \sec^{2k} x \, dx \\
&= \int \tan^m x \sec^{2k-2} x \sec^2 x \, dx \\
&= \int \tan^m x \sec^{2(k-1)} x \sec^2 x \, dx \\
&= \int \tan^m x \left(\sec^2 x\right)^{k-1} \sec^2 x \, dx \\
&= \int \tan^m x \left(1 + \tan^2 x\right)^{k-1} \sec^2 x \, dx \\
&\stackrel{CV}{=} \int u^m \left(1 + u^2\right)^{k-1} \, du \\
&= (\text{Expand, then Distribute, then Integrate})
\end{aligned}$$

CV: Let $u = \tan x \implies du = \sec^2 x \, dx \implies \sec^2 x \, dx = du$

Trig Integrals involving Tangent & Secant (CASE II)

TASK: Evaluate $I = \displaystyle\int \tan^m x \sec^n x \, dx$, where $m, n \in \bar{\mathbb{N}} := \{0, 1, 2, 3, \cdots\}$

CASE II: m is odd AND $n \neq 0 \implies m = 2k+1$, where $k \in \bar{\mathbb{N}}$

$$\begin{aligned}
I &= \int \tan^{2k+1} x \sec^n x \, dx \\
&= \int \tan^{2k} x \sec^{n-1} x \sec x \tan x \, dx \\
&= \int \left(\tan^2 x\right)^k \sec^{n-1} x \sec x \tan x \, dx \\
&= \int \left(\sec^2 x - 1\right)^k \sec^{n-1} x \sec x \tan x \, dx \\
&\overset{CV}{=} \int \left(u^2 - 1\right)^k u^{n-1} \, du \\
&= \text{(Expand, then Distribute, then Integrate)}
\end{aligned}$$

CV: Let $u = \sec x \implies du = \sec x \tan x \, dx \implies \sec x \tan x \, dx = du$

Trig Integrals involving Tangent & Secant (CASE X)

TASK: Evaluate $I = \int \tan^m x \sec^n x \, dx$, where $m, n \in \bar{\mathbb{N}} := \{0, 1, 2, 3, \cdots\}$

CASE X: All other cases

$$\int \tan^3 x \, dx, \int \tan^5 x \, dx, \int \sec^3 x \, dx, \int \sec^5 x \, dx, \int \tan^2 x \sec x \, dx,$$
$$\int \tan^2 x \sec^3 x \, dx, \int \tan^4 x \sec^3 x \, dx, \int \tan^4 x \sec^5 x \, dx, \int \tan^{12} x \sec^{19} x \, dx, \cdots$$

- Requires clever use of:

  trig identities, CV, IBP, $\int \tan x \, dx$, and/or $\int \sec x \, dx$

- For $n \geq 5$, use **reduction formula**:

$$\int \sec^n(\alpha u) \, du = \frac{\sec^{n-2}(\alpha u) \tan(\alpha u)}{\alpha(n-1)} + \frac{n-2}{n-1} \int \sec^{n-2}(\alpha u) \, du$$

where $\alpha \in \mathbb{R}$

Trig Integrals involving Cotangent & Cosecant (Toolkit)

TASK: Evaluate $I = \displaystyle\int \cot^m x \csc^n x \, dx,$ where $m, n \in \bar{\mathbb{N}} := \{0, 1, 2, 3, \cdots\}$

- Binomial Theorem: $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^n$
- Pascal's Triangle
- Exponent Rules: $x^m x^n = x^{m+n}$, $(x^m)^n = x^{mn}$
- Pythagorean Identity: $1 + \cot^2 x = \csc^2 x$
- Quotient Identity: $\cot x = \frac{\cos x}{\sin x}$
- Negative-Angle Identities: $\cot(-x) = -\cot x$, $\csc(-x) = -\csc x$
- Change of Variables (CV)
- Integration by Parts (IBP)
- $\int u^n \, du = \frac{1}{n+1} u^{n+1} + C$
- $\int \cot x \, dx = \ln|\sin x| + C$
- $\int \csc x \, dx = -\ln|\csc x + \cot x| + C$

Trig Integrals involving Cot & Csc (CASE I)

TASK: Evaluate $I = \int \cot^m x \csc^n x \, dx$, where $m, n \in \bar{\mathbb{N}} := \{0, 1, 2, 3, \cdots\}$

CASE I: n is even $\implies$ $n = 2k$, where $k \in \mathbb{N}$

$$\begin{aligned} I &= \int \cot^m x \csc^{2k} x \, dx \\ &= \int \cot^m x \csc^{2k-2} x \csc^2 x \, dx \\ &= \int \cot^m x \csc^{2(k-1)} x \csc^2 x \, dx \\ &= \int \cot^m x \left(\csc^2 x\right)^{k-1} \csc^2 x \, dx \\ &= \int \cot^m x \left(1 + \cot^2 x\right)^{k-1} \csc^2 x \, dx \\ &\overset{CV}{=} \int u^m \left(1 + u^2\right)^{k-1} (-du) \\ &= \left(\text{Expand, then Distribute, then Integrate}\right) \end{aligned}$$

CV: Let $u = \cot x \implies du = -\csc^2 x \, dx \implies \csc^2 x \, dx = -du$

Trig Integrals involving Cot & Csc (CASE II)

TASK: Evaluate $I = \int \cot^m x \csc^n x \, dx$, where $m, n \in \bar{\mathbb{N}} := \{0, 1, 2, 3, \cdots\}$

CASE II: m is odd AND $n \neq 0 \implies m = 2k+1$, where $k \in \bar{\mathbb{N}}$

$$
\begin{aligned}
I &= \int \cot^{2k+1} x \csc^n x \, dx \\
&= \int \cot^{2k} x \csc^{n-1} x \csc x \cot x \, dx \\
&= \int \left(\cot^2 x\right)^k \csc^{n-1} x \csc x \cot x \, dx \\
&= \int \left(\csc^2 x - 1\right)^k \csc^{n-1} x \csc x \cot x \, dx \\
&\overset{CV}{=} \int \left(u^2 - 1\right)^k u^{n-1} (-du) \\
&= \text{(Expand, then Distribute, then Integrate)}
\end{aligned}
$$

CV: Let $u = \csc x \implies du = -\csc x \cot x \, dx \implies \csc x \cot x \, dx = -du$

Trig Integrals involving Cot & Csc (CASE X)

TASK: Evaluate $I = \int \cot^m x \csc^n x \, dx$, where $m, n \in \bar{\mathbb{N}} := \{0, 1, 2, 3, \cdots\}$

CASE X: All other cases

$$\int \cot^3 x \, dx, \int \cot^5 x \, dx, \int \csc^3 x \, dx, \int \csc^5 x \, dx, \int \cot^2 x \csc x \, dx,$$
$$\int \cot^2 x \csc^3 x \, dx, \int \cot^4 x \csc^3 x \, dx, \int \cot^4 x \csc^5 x \, dx, \int \cot^{12} x \csc^{19} x \, dx, \cdots$$

- Requires clever use of:

  trig identities, CV, IBP, $\int \cot x \, dx$, and/or $\int \csc x \, dx$

- For $n \geq 5$, use **reduction formula**:

$$\int \csc^n(\alpha u) \, du = -\frac{\csc^{n-2}(\alpha u)\cot(\alpha u)}{\alpha(n-1)} + \frac{n-2}{n-1}\int \csc^{n-2}(\alpha u) \, du$$

where $\alpha \in \mathbb{R}$

Trig Integrals: Sine & Cosine with Different Arguments

Let $m \neq n$:

$$\int \sin(mx)\cos(nx)\,dx = \frac{1}{2}\int \sin\Big[(m+n)x\Big]\,dx + \frac{1}{2}\int \sin\Big[(m-n)x\Big]\,dx$$

$$\int \cos(mx)\cos(nx)\,dx = \frac{1}{2}\int \cos\Big[(m+n)x\Big]\,dx + \frac{1}{2}\int \cos\Big[(m-n)x\Big]\,dx$$

$$\int \sin(mx)\sin(nx)\,dx = -\frac{1}{2}\int \cos\Big[(m+n)x\Big]\,dx + \frac{1}{2}\int \cos\Big[(m-n)x\Big]\,dx$$

PART II:

INTEGRALS REQUIRING TRIG SUBSTITUTION

Integrals involving $\sqrt{a^2 - u^2}$

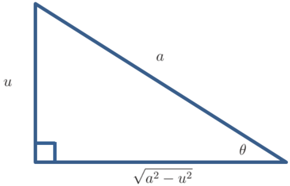
(Here, $a > 0$ and $k \in \bar{\mathbb{N}} := \{0, 1, 2, 3, \cdots\}$)

$$\int \sqrt{a^2 - u^2}\, du, \quad \int u^{2k+1}\sqrt{a^2 - u^2}\, du, \quad \int \frac{1}{\sqrt{a^2 - u^2}}\, du,$$

$$\int \frac{u^{2k+1}}{\sqrt{a^2 - u^2}}\, du, \quad \int \frac{1}{u^{2k}\sqrt{a^2 - u^2}}\, du, \quad \int \frac{\sqrt{a^2 - u^2}}{u^{2k}}\, du$$

- CV: Let $u = a\sin\theta \iff \sin\theta = \frac{u}{a} \equiv \frac{\text{OPP}}{\text{HYP}} \iff \theta = \arcsin\left(\frac{u}{a}\right)$

  Then: $du = a\cos\theta\, d\theta$ and $\sqrt{a^2 - u^2} = a\cos\theta$
- After simplifying, the resulting integral is often a **trig integral**.
- Build an appropriate **reference triangle** to help with **back substitution**.



Integrals involving $\sqrt{a^2+u^2}$

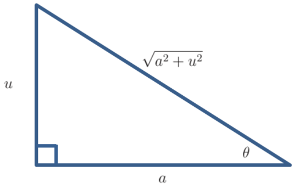
$$\Big(\text{Here, } a > 0 \text{ and } k \in \bar{\mathbb{N}} := \{0,1,2,3,\cdots\}\Big)$$

$$\int \sqrt{a^2+u^2}\,du,\quad \int u^{2k+1}\sqrt{a^2+u^2}\,du,\quad \int \frac{1}{\sqrt{a^2+u^2}}\,du,$$

$$\int \frac{u^{2k+1}}{\sqrt{a^2+u^2}}\,du,\quad \int \frac{1}{u^{2k}\sqrt{a^2+u^2}}\,du,\quad \int \frac{\sqrt{a^2+u^2}}{u^{2k+4}}\,du$$

- CV: Let $u = a\tan\theta \iff \tan\theta = \frac{u}{a} \equiv \frac{\text{OPP}}{\text{ADJ}} \iff \theta = \arctan\left(\frac{u}{a}\right)$

  Then: $du = a\sec^2\theta\,d\theta$ and $\sqrt{a^2+u^2} = a\sec\theta$
- After simplifying, the resulting integral is often a **trig integral**.
- Build an appropriate **reference triangle** to help with **back substitution**.



Integrals involving $\sqrt{u^2 - a^2}$

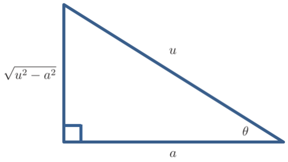
(Here, $a > 0$ and $k \in \bar{\mathbb{N}} := \{0, 1, 2, 3, \cdots\}$)

$$\int \sqrt{u^2 - a^2}\, du, \quad \int u^{2k+1}\sqrt{u^2 - a^2}\, du, \quad \int \frac{1}{\sqrt{u^2 - a^2}}\, du,$$

$$\int \frac{u^{2k+1}}{\sqrt{u^2 - a^2}}\, du, \quad \int \frac{1}{u^{2k}\sqrt{u^2 - a^2}}\, du, \quad \int \frac{\sqrt{u^2 - a^2}}{u^{2k}}\, du$$

- CV: Let $u = a\sec\theta \iff \sec\theta = \frac{u}{a} \equiv \frac{\text{HYP}}{\text{ADJ}} \iff \theta = \text{arcsec}\left(\frac{u}{a}\right)$

  Then: $du = a\sec\theta\tan\theta\, d\theta$ and $\sqrt{u^2 - a^2} = a\tan\theta$
- After simplifying, the resulting integral is often a **trig integral**.
- Build an appropriate **reference triangle** to help with **back substitution**.



Integrals involving $\sqrt{x^2+bx+c}$

- 1^{st}, **complete the square (CS)**.
- 2^{nd}, **change variables (CV)**.
- Now, the integral involves $\sqrt{u^2-a^2}$, $\sqrt{u^2+a^2}$, or $\sqrt{a^2-u^2}$, so use **trig substitution**.

- $\displaystyle\int \sqrt{x^2+5x+8}\,dx \overset{CS}{=} \int \sqrt{\left(x+\tfrac{5}{2}\right)^2+\left(\tfrac{\sqrt{7}}{2}\right)^2}\,dx \overset{CV}{=} \int \sqrt{u^2+a^2}\,du$
- $\displaystyle\int \sqrt{3x^2-5x-8}\,dx \overset{CS}{=} \int \sqrt{3\left[\left(x-\tfrac{5}{6}\right)^2+\left(\tfrac{11}{6}\right)^2\right]}\,dx \overset{CV}{=} \int \sqrt{3}\sqrt{u^2-a^2}\,du$
- $\displaystyle\int \frac{1}{\sqrt{2-2x-x^2}}\,dx \overset{CS}{=} \int \frac{1}{\sqrt{\left(\sqrt{3}\right)^2-(x+1)^2}}\,dx \overset{CV}{=} \int \frac{1}{\sqrt{a^2-u^2}}\,du$

Integrals of the form $\int \frac{1}{x^2+bx+c}\, dx$

- 1^{st}, **complete the square (CS)**.
- 2^{nd}, **change variables (CV)**.
- Now, the integral involves $u^2 - a^2,\ u^2 + a^2,$ or $a^2 - u^2$, so use **trig substitution**.

REMARK: If $ax^2 + bx + c$ can be easily factored as a product of real linear factors, the next integration technique may be easier to use.

Fin.