Partial Fraction Decomposition (PFD)

Calculus II

Josh Engwer

TTU

05 March 2014

Degree of a Polynomial

Recall from Algebra the **degree** of a polynomial:

Definition

The **degree** of a polynomial is the power of its highest-power term.

$$
\begin{aligned}
\text{deg}\left[7x^5 - x^4 + 2x^3 + 11x^2 - 8x + 17\right] &= 5\\
\text{deg}\left[16 - x - x^7 + 3x^9 + x^{12}\right] &= 12\\
\text{deg}\left[x^4 - x^2 - 1\right] &= 4\\
\text{deg}\left[-8 - x^3\right] &= 3\\
\text{deg}\left[3x^2 - 5x + 7\right] &= 2\\
\text{deg}\left[6 - 2x\right] &= 1\\
\text{deg}\left[37\right] &= 0
\end{aligned}
$$

REMARK:

Negative powers are not polynomials: $1 - 3x^{-1} + 5x^{-2} = 1 - \dfrac{3}{x} + \dfrac{5}{x^2}$

Factoring Polynomials (Review)

Monomial Factoring:	$8x^6 + 2x^4$	$=$	$2x^4\left(4x^2+1\right)$
Factoring by Grouping:	$x^3 + 3x^2 + 9x + 27$	$=$	$\left(x^2+9\right)(x+3)$
Difference of Squares:	$x^2 - y^2$	$=$	$(x+y)(x-y)$
Difference of Cubes:	$x^3 - y^3$	$=$	$(x-y)\left(x^2+xy+y^2\right)$
Sum of Cubes:	$x^3 + y^3$	$=$	$(x+y)\left(x^2-xy+y^2\right)$

REMARK: All 5^{th}-degree or higher polynomials will be factored a priori.

Irreducible Quadratics

Definition

Let $a, b, c \in \mathbb{R}$. Then:

The **discriminant** of quadratic $ax^2 + bx + c$ is defined to be $b^2 - 4ac$.

Definition

Let $a, b, c \in \mathbb{R}$. Then:

Quadratic $ax^2 + bx + c$ is an **irreducible quadratic** $\iff b^2 - 4ac < 0$.

i.e., the linear factors of an irreducible quadratic are **complex** (not real):

(Recall that the **imaginary number** $i = \sqrt{-1}$.)

- $x^2 + 1$ is irreducible since $x^2 + 1 = (x - i)(x + i)$ $\quad \left[b^2 - 4ac = -4 < 0\right]$
- $x^2 - 1$ is reducible since $x^2 - 1 = (x - 1)(x + 1)$ $\quad \left[b^2 - 4ac = \quad 4 > 0\right]$
- $x^2 + 2x + 2$ is irreducible since $x^2 + 2x + 2 = [x + (1 - i)][x + (1 + i)]$ $\left[b^2 - 4ac = -4 < 0\right]$

Fundamental Theorem of Algebra (FTA)

Theorem

(Fundamental Theorem of Algebra)

*Every n^{th}-degree **polynomial with complex coefficients** can be factored into n **linear factors with complex coefficients**, some of which may be repeated.*

Corollary

*Every n^{th}-degree **polynomial with real coefficients** can be factored into **linears & irreducible quadratics with real coefficients**.*

REMARK: The FTA provides no procedure for factoring!

- $x^4 + 1 = \left(x^2 + \sqrt{2}x + 1\right)\left(x^2 - \sqrt{2}x + 1\right)$
- $x^5 - 1 = (x-1)\left(x^2 + \frac{1+\sqrt{5}}{2}x + 1\right)\left(x^2 + \frac{1-\sqrt{5}}{2}x + 1\right)$
- $x^5 + 1 = (x+1)\left(x^2 - \frac{1+\sqrt{5}}{2}x + 1\right)\left(x^2 - \frac{1-\sqrt{5}}{2}x + 1\right)$
- $x^6 - 1 = (x-1)(x+1)\left(x^2 + x + 1\right)\left(x^2 - x + 1\right)$
- $x^6 + 1 = \left(x^2 + 1\right)\left(x^2 + \sqrt{3}x + 1\right)\left(x^2 - \sqrt{3}x + 1\right)$

Integration of Rational Functions & Roots

Partial Fraction Decomposition is used to integrate **rational fcns** & **roots**:

$$\int \frac{dx}{(x-1)(x+7)}, \quad \int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}, \quad \int \frac{1+\sqrt{x}}{x^2+1}\,dx, \quad \int \frac{\cos x}{\sin^2 x + \sin x}\,dx, \ \ldots$$

Polynomial Division

Proposition

Let $N(x), D(x)$ *be polynomials such that deg*$[N(x)] \geq$ *deg*$[D(x)]$. *Then:*

$$\frac{N(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$$

where $Q(x), R(x)$ *are polynomials s.t. deg*$[R(x)] <$ *deg*$[D(x)]$

Polynomial Division (Example)

WORKED EXAMPLE: Simplify $\dfrac{x^5+4x^3-x^2-6}{x^3-x+4}$.

$$
\begin{array}{r|l}
 & \qquad\qquad\qquad\qquad x^2+0x+5 \\
\hline
x^3+0x^2-x+4 & x^5+0x^4+4x^3-x^2+0x-6 \\
 & \underline{-x^5-0x^4+x^3-4x^2} \\
 & \qquad\qquad 5x^3-5x^2+0x \\
 & \qquad \underline{-0x^4-0x^3-0x^2-0x} \\
 & \qquad\qquad 5x^3-5x^2+0x-6 \\
 & \qquad\quad \underline{-5x^3-0x^2+5x-20} \\
 & \qquad\qquad\qquad -5x^2+5x-26
\end{array}
$$

$$\therefore\ \frac{x^5+4x^3-x^2-6}{x^3-x+4} = \boxed{x^2+5+\frac{-5x^2+5x-26}{x^3-x+4}}$$

PFD (Distinct Factors)

Let $p_1, p_2, q_1, q_2, a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$. Then:

$$\frac{1}{(p_1x+q_1)(p_2x+q_2)} \overset{PFD}{=} \frac{A_1}{p_1x+q_1} + \frac{A_2}{p_2x+q_2}$$

$$\frac{1}{(a_1x^2+b_1x+c_1)\,(a_2x^2+b_2x+c_2)} \overset{PFD}{=} \frac{B_1x+C_1}{a_1x^2+b_1x+c_1} + \frac{B_2x+C_2}{a_2x^2+b_2x+c_2}$$

where $A_1, A_2, B_1, B_2, C_1, C_2 \in \mathbb{R}$

WORKED EXAMPLE: $\frac{1-3x}{x^2-5x+6} = \frac{1-3x}{(x-2)(x-3)} \overset{PFD}{=} \frac{A}{x-2} + \frac{B}{x-3}$

WORKED EXAMPLE: $\frac{x^3+2x^2-7}{x(x-1)\,(x^2+7)} \overset{PFD}{=} \frac{A}{x} + \frac{B}{x-1} + \frac{\widehat{C}x+D}{x^2+7}$

PFD (Repeated Factors)

Let $p, q \in \mathbb{R}$ and $m, n \in \mathbb{N}$. Then:

$$\frac{1}{(px+q)^m} \overset{PFD}{=} \frac{A_1}{px+q} + \frac{A_2}{(px+q)^2} + \frac{A_3}{(px+q)^3} + \cdots + \frac{A_m}{(px+q)^m}$$

$$\frac{1}{\left(ax^2+bx+c\right)^n} \overset{PFD}{=} \frac{B_1x+C_1}{ax^2+bx+c} + \frac{B_2x+C_2}{\left(ax^2+bx+c\right)^2} + \cdots + \frac{B_nx+C_n}{\left(ax^2+bx+c\right)^n}$$

where $A_1, A_2, \ldots, A_m \in \mathbb{R}$ $\quad B_1, B_2, \ldots, B_n \in \mathbb{R}$ $\quad C_1, C_2, \ldots, C_n \in \mathbb{R}$

WORKED EXAMPLE: $\frac{9}{(2x+5)^3} \overset{PFD}{=} \frac{A}{2x+5} + \frac{B}{(2x+5)^2} + \frac{\widehat{C}}{(2x+5)^3}$

WORKED EXAMPLE: $\frac{1}{\left(x^2-x+5\right)^2} \overset{PFD}{=} \frac{Ax+B}{x^2-x+5} + \frac{\widehat{C}x+D}{\left(x^2-x+5\right)^2}$

PFD (Example)

WORKED EXAMPLE: Compute $I = \int \frac{1-3x}{(x-2)(x-3)}\, dx$.

$$\frac{1-3x}{(x-2)(x-3)} \overset{PFD}{=} \frac{A}{x-2} + \frac{B}{x-3}$$

Multiply both sides by the denominator: $1-3x = A(x-3) + B(x-2)$

Distribute terms: $1-3x = Ax - 3A + Bx - 2B$

Collect like terms: $-3x + 1 = (A+B)x + (-3A-2B)$

Equate like terms: $\begin{cases} A + B = -3 \\ -3A - 2B = 1 \end{cases}$

Solve linear system: $A = 5,\ B = -8$

$$\therefore \int \frac{1-3x}{(x-2)(x-3)}\, dx \overset{PFD}{=} \int \frac{5}{x-2}\, dx - \int \frac{8}{x-3}\, dx$$

$$= \boxed{5\ln|x-2| - 8\ln|x-3| + C} = \ln\left|\frac{(x-2)^5}{(x-3)^8}\right| + C$$

PFD (Procedure)

TASK: Compute $I = \int \frac{N(x)}{D(x)}\,dx$, where $N(x), D(x)$ are polynomials.

- STEP 1: If $\deg[N(x)] \geq \deg[D(x)]$, then perform **polynomial division**.
- STEP 2: Factor denominator $D(x)$ into linear factors & irreducible quadratics.
- STEP 3: Write out the PFD with the unknown numerators.
- STEP 4: Multiply both sides by the denominator.
- STEP 5: Distribute all terms.
- STEP 6: Collect all like terms.
- STEP 7: Equate all like terms.
- STEP 8: Solve resulting linear system.
- STEP 9: The resulting integrals are now simple(r) to work with.
    - REMARK: A **change of variables (CV)** or **completing the square (CS)** may be necessary.

Fin.