Improper Integrals

Calculus II

Josh Engwer

TTU

10 March 2014

Limits of Functions (Toolkit)

TASK: Evaluate limit (if it exists): $\lim_{x \to x_0} f(x)$

READ: "The limit as x approaches x_0 of $f(x)$"

- Naïve Substitution (NS)
- Factor Polynomial(s) & cancel like factors
- Rationalize Numerator/Denominator (RN/RD)
- Combine Fractions
- Throw a factor downstairs (TFD)
- Change of Variables (CV): Let $u = g(x)$, then limit becomes $\lim_{u \to g(x_0)} h(u)$
- Apply a Trig Identity
- L'Hôpital's Rule (LHOP)
- Find One-Sided Limits first
- Sketch the Function
- Squeeze Theorem
- Build a Table – TOO INACCURATE!
- δ-ϵ definition of limit – TOO HARD!

Limits of Functions (Infinity)

- Remember, ∞ is **not a real number**, but rather a **symbol** indicating **growth without bound**.
- Similarly, $-\infty$ indicates **decay without bound.**
- However, $\pm\infty$ satisfy some arithmetic properties that agree with intuition $(x \in \mathbb{R}, n \in \mathbb{N})$:
  - $\infty + \infty = \infty \qquad -\infty - \infty = -\infty$
  - $\infty + x = x + \infty = \infty$
  - $-\infty + x = x - \infty = -\infty$
  - $(\infty)(\infty) = \infty, \qquad (-\infty)(-\infty) = \infty$
  - $(-\infty)(\infty) = -\infty, \qquad (\infty)(-\infty) = -\infty$
  - $x > 0 \implies x \cdot \infty = \infty$ and $x \cdot (-\infty) = -\infty$
  - $x < 0 \implies x \cdot \infty = -\infty$ and $x \cdot (-\infty) = \infty$
  - $\infty^n = \infty \qquad \sqrt[n]{\infty} = \infty$

Limits of Functions (Indeterminant Forms)

Recall the seven **indeterminant forms** from Single Variable Calculus:

$$\frac{0}{0}, \frac{\infty}{\infty}, 0\cdot\infty, \infty-\infty, 0^0, \infty^0, 1^\infty$$

Remember that **L'Hôpital's Rule (LHOP)** only works with $\frac{0}{0}$ or $\frac{\infty}{\infty}$:

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} \overset{NS}{=} \frac{0}{0} \overset{LHOP}{=} \lim_{x\to x_0}\frac{f'(x)}{g'(x)}$$

$$\lim_{x\to x_0}\frac{f(x)}{g(x)} \overset{NS}{=} \frac{\infty}{\infty} \overset{LHOP}{=} \lim_{x\to x_0}\frac{f'(x)}{g'(x)}$$

Convert $0^0, \infty^0, 1^\infty$ to $0\cdot\infty$ by: Taking the natural logarithm: $\ln A^k = k\ln A$

Convert $0\cdot\infty$ to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by: Throwing a factor downstairs/upstairs

Convert $\infty-\infty$ to $\frac{\infty}{\infty}$ by: Combining fractions or Rationalizing numerator

Limits of Functions (Naïve Substitution)

WORKED EXAMPLE: Evaluate $\lim_{t\to4}\sqrt{5+t}$.

$$\lim_{t\to4}\sqrt{5+t}\overset{NS}{=}\sqrt{5+(4)}=\sqrt{9}=\boxed{3}$$

WORKED EXAMPLE: Evaluate $\lim_{x\to\infty}\left(x^{2/3}-7\right)$.

$$\lim_{x\to\infty}\left(x^{2/3}-7\right)\overset{NS}{=}(\infty)^{2/3}-7=\infty-7=\boxed{\infty}$$

WORKED EXAMPLE: Evaluate $\lim_{x\to0^+}x^2\ln x$.

$$\lim_{x\to0^+}x^2\ln x\overset{NS}{=}0\cdot(-\infty)=0\cdot\infty \implies \text{Rewrite/Simplify Function}$$

$$\overset{TFD}{=}\lim_{x\to0^+}\frac{\ln x}{1/x^2}\overset{NS}{=}\frac{-\infty}{\infty}\overset{LHOP}{=}\lim_{x\to0^+}\frac{1/x}{-2/x^3}=\lim_{x\to0^+}-\frac{1}{2}x^2\overset{NS}{=}-\frac{1}{2}(0)^2=\boxed{0}$$

Unbounded Intervals

Recall from Calculus I the definitions **bounded & unbounded intervals**:

Definition

The following are all **bounded intervals**:

Open Interval	(a, b)
Closed Interval	$[a, b]$
Half-open Intervals	$(a, b], [a, b)$

Definition

The following are all **unbounded intervals**:

$$[a, \infty), (a, \infty), (-\infty, b], (-\infty, b), (-\infty, \infty)$$

REMARK: The **real line** $\mathbb{R} := (-\infty, \infty)$

Break Discontinuities

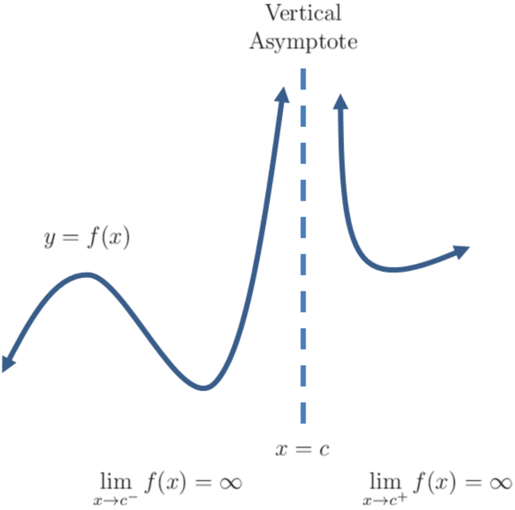
Recall from Calculus I the definition of a **break discontinuity**:

Definition

A function f has a **break discontinuity** at $x = c$ if at least one 1-sided limit is infinite:

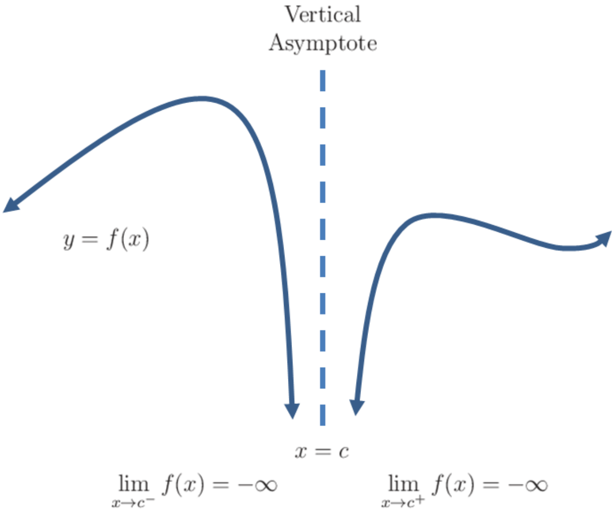
$$\left[\lim_{x\to c^-} f(x) = -\infty \text{ or } \infty\right] \text{ AND/OR } \left[\lim_{x\to c^+} f(x) = -\infty \text{ or } \infty\right]$$

Break Discontinuities (Example)



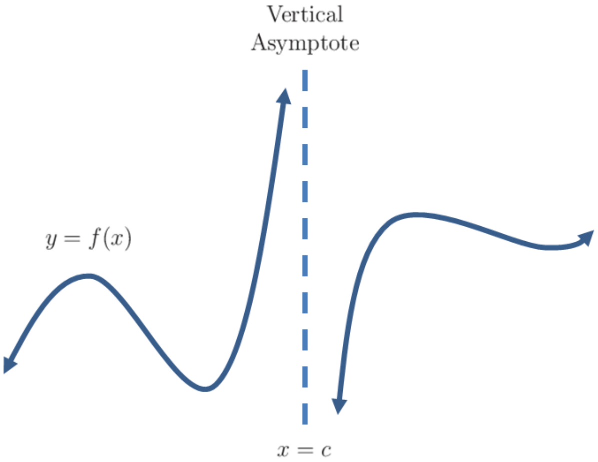
Break Discontinuity at $x = c$

Break Discontinuities (Example)



Break Discontinuity at $x = c$

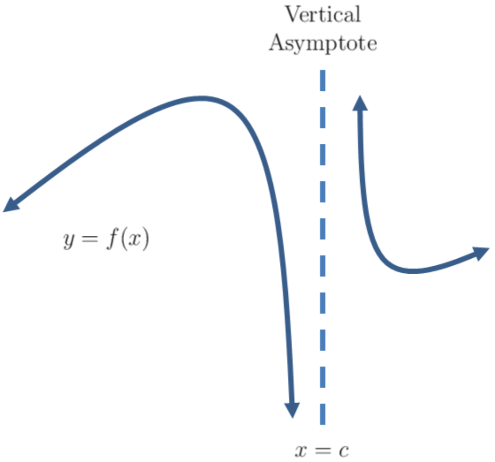
Break Discontinuities (Example)



$$\lim_{x \to c^-} f(x) = \infty \qquad \lim_{x \to c^+} f(x) = -\infty$$

Break Discontinuity at $x = c$

Break Discontinuities (Example)



$$\lim_{x \to c^-} f(x) = -\infty \qquad \lim_{x \to c^+} f(x) = \infty$$

Break Discontinuity at $x = c$

Recall from Calculus I the **Fundamental Theorem of Calculus (FTC)**:

Theorem

Let function $f \in C^1[a,b]$. *Then*

$$\int_a^b f'(x)\,dx = f(b) - f(a)$$

Notice that interval $[a,b]$ is **bounded** & f is **continuous** on $[a,b]$.

QUESTION: What if....

- .... the interval is **unbounded**??
- .... the function has **break discontinuities**??

ANSWER: The resulting integral is **improper**.

Improper Integrals (Unbounded Intervals)

Proposition

$$f \in C[a,\infty) \implies \int_a^\infty f(x)\,dx := \lim_{B\to\infty} \int_a^B f(x)\,dx$$

Proposition

$$f \in C(-\infty,b] \implies \int_{-\infty}^b f(x)\,dx := \lim_{A\to-\infty} \int_A^b f(x)\,dx$$

Proposition

$$f \in C(\mathbb{R}) \implies \int_{-\infty}^\infty f(x)\,dx := \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx, \qquad \textit{where } c \in \mathbb{R}$$

REMARK: The **improper integral converges** if the corresponding **limit exists and is finite.** Otherwise, it **diverges.**

WARNING: In general, $\int_{-\infty}^\infty f(x)\,dx \neq \lim_{L\to\infty} \int_{-L}^L f(x)\,dx$

Improper Integrals (Break Discontinuities)

Proposition

Let $f \in C[a,b)$ s.t. f has a ***break discontinuity*** *at $x = b$. Then*

$$\int_a^b f(x)\,dx := \lim_{B \to b^-} \int_a^B f(x)\,dx$$

Proposition

Let $f \in C(a,b]$ s.t. f has a ***break discontinuity*** *at $x = a$. Then*

$$\int_a^b f(x)\,dx := \lim_{A \to a^+} \int_A^b f(x)\,dx$$

Proposition

Let $f \in C[a,b]$ s.t. f has a ***break discontinuity*** *at $x = c \in (a,b)$. Then*

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Special Limits involving Infinity

$$\lim_{x\to\infty} x^n = \lim_{x\to\infty} \sqrt[n]{x} = \infty \qquad (n \in \mathbb{N})$$

$$\lim_{x\to\infty} \frac{1}{x^n} = \lim_{x\to\infty} \frac{1}{\sqrt[n]{x}} = 0 \qquad (n \in \mathbb{N})$$

$$\lim_{x\to\infty} \ln x = \lim_{x\to\infty} \log x = \lim_{x\to\infty} \log_2 x = \infty$$

$$\lim_{x\to 0^+} \ln x = \lim_{x\to 0^+} \log x = \lim_{x\to 0^+} \log_2 x = -\infty$$

$$\lim_{x\to\infty} e^x = \lim_{x\to\infty} 10^x = \lim_{x\to\infty} 2^x = \infty$$

$$\lim_{x\to -\infty} e^x = \lim_{x\to -\infty} 10^x = \lim_{x\to -\infty} 2^x = 0$$

$$\lim_{x\to\infty} \arctan x = \pi/2$$

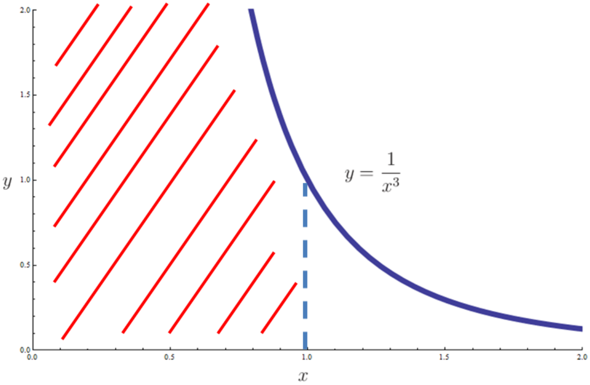
$$\lim_{x\to -\infty} \arctan x = -\pi/2$$

Improper Integrals (Example)

WORKED EXAMPLE: Evaluate $I = \int_0^1 \frac{1}{x^3}\, dx$.

$$I = \lim_{A\to 0+} \int_A^1 x^{-3}\, dx = \lim_{A\to 0+} \left[-\frac{1}{2x^2}\right]_{x=A}^{x=1} \overset{FTC}{=} \lim_{A\to 0+} \left[-\frac{1}{2} + \frac{1}{2A^2}\right] = -\frac{1}{2} + \infty = \boxed{\infty}$$

$\therefore$ The region below in red is **unbounded** and **infinite**:

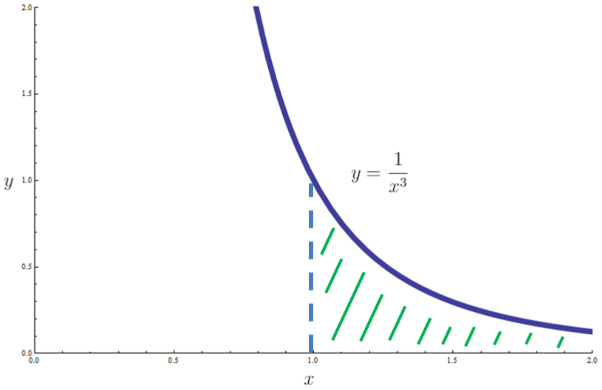


Improper Integrals (Example)

WORKED EXAMPLE: Evaluate $I = \int_1^{\infty} \frac{1}{x^3}\, dx$.

$$I = \lim_{B\to\infty} \int_1^B x^{-3}\, dx = \lim_{B\to\infty} \left[-\frac{1}{2x^2}\right]_{x=1}^{x=B} \overset{FTC}{=} \lim_{B\to\infty} \left[\frac{1}{2} - \frac{1}{2B^2}\right] = \frac{1}{2} - 0 = \boxed{\frac{1}{2}}$$

$\therefore$ The region below in green is **unbounded** yet **finite**:



Improper Integrals (Example)

WORKED EXAMPLE: Compute $I = \int_{e^2}^{\infty} \frac{1}{x \ln x}\, dx$.

$$I = \lim_{B\to\infty} \int_{e^2}^{B} \frac{1}{x \ln x}\, dx$$

CV: Let $u = \ln x$. Then $du = \frac{1}{x}\, dx$ and $u(B) = \ln B, u\left(e^2\right) = \ln\left(e^2\right) = 2$

$$\overset{CV}{=} \lim_{B\to\infty} \int_{2}^{\ln B} \frac{1}{u}\, du = \lim_{B\to\infty} \Big[\ln|u| \Big]_{u=2}^{u=\ln B}$$

$$\overset{FTC}{=} \lim_{B\to\infty} \Big[\ln(\ln B) - \ln 2 \Big] = \infty - \ln 2 = \boxed{\infty}$$

(i.e. the integral **diverges**)

Why is $\lim_{B\to\infty} \ln(\ln B) = \infty$??

CV: Let $B^* = \ln B$. Then, $B \to \infty \implies \ln B \to \infty \implies B^* \to \infty$

$$\therefore \quad \lim_{B\to\infty} \ln(\ln B) \overset{CV}{=} \lim_{B^*\to\infty} \ln B^* = \infty$$

Special Functions

Some **special functions** are defined by improper integrals:

- Gamma Function: $\Gamma(\alpha) := \int_0^\infty x^{\alpha-1} e^{-x}\, dx$, where $\alpha > 0$
- Complementary Error Function: $\text{erfc}(x) := \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-t^2}\, dt$
- Airy Function (of the 1^{st} kind): $\text{Ai}(x) := \frac{1}{\pi} \int_0^\infty \cos\left(\frac{t^3}{3} + xt\right) dt$
- Airy Function (of the 2^{nd} kind):

$$\text{Bi}(x) := \frac{1}{\pi} \int_0^\infty \left[\exp\left(-\frac{t^3}{3} + xt\right) + \sin\left(\frac{t^3}{3} + xt\right)\right] dt$$

Special functions are ”special” in the sense that they tend to show up often in certain branches of mathematics, statistics, physics, and engineering.

IMPORTANT: DO NOT MEMORIZE THESE SPECIAL FUNCTIONS!

Integral Transforms

Improper integrals show up in **integral transforms**:

- Laplace Transform: $F(s) = \mathcal{L}\{f(t)\} := \int_0^\infty e^{-st} f(t)\, dt$
- Fourier Transform: $\widehat{f}(\omega) = \mathcal{F}\{f(t)\} := \int_{-\infty}^\infty f(t) e^{-i\omega t}\, dt$

Real Number $\rightarrow$ Function $\rightarrow$ Real Number

Function of x $\rightarrow$ Function $\rightarrow$ Function of x

Function of t $\rightarrow$ Laplace Transform $\rightarrow$ Function of s

Function of time t $\rightarrow$ Fourier Transform $\rightarrow$ Function of frequency ω

Integral transforms are instrumental for certain **differential equations**.

IMPORTANT: Integral transforms are beyond the scope of this course.

Fin

Fin.