Sequences

Calculus II

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TTU

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Sequences (Definition)

$\mathbb{Z} \equiv$ set of **integers** $\quad \mathbb{R} \equiv$ set of **real numbers** $\quad \mathbb{N} := \{1, 2, 3, 4, \cdots\}$

So far in Calculus, a **function** $f(x)$ had its domain & range as follows:

$$\text{Dom}(f) \subset \mathbb{R} \text{ \& } \text{Rng}(f) \subset \mathbb{R}$$

What if the domain of a function is **restricted to integers**?

Definition

A **sequence** $\{a_n\}$, is a function s.t. $\text{Dom}(a_n) \subset \mathbb{Z}$ & $\text{Rng}(a_n) \subset \mathbb{R}$.

NOTATION: Sequence is denoted $\{a_n\}_{n=1}^{\infty}$ or $\{a_n\}$ or $(a_1, a_2, a_3, a_4, a_5, \cdots)$.

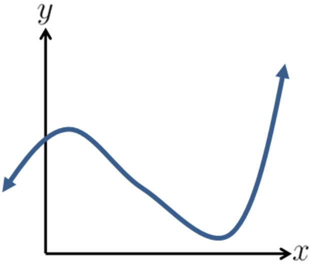
$n \in \mathbb{N}$ is called the **index** of the sequence (the label of which doesn't matter).

WORKED EXAMPLE: $\left\{\frac{1}{n^2}\right\} = \left(\frac{1}{1^2}, \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{4^2}, \cdots\right) = \left(1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \cdots\right)$

WORKED EXAMPLE: $\{n^2 + 1\}_{n=-1}^{\infty} = (2, 1, 2, 5, 10, 17, \cdots)$

WORKED EXAMPLE: $\{j^2 + 1\}_{j=-1}^{\infty} = (2, 1, 2, 5, 10, 17, \cdots)$

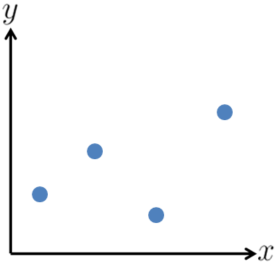
Sequences (Plot)



Function $f(x)$

Curve

$\mathrm{Dom}(f) \subset \mathbb{R}$



Sequence $\{a_n\}$

Set of Points

$\mathrm{Dom}(a_n) \subset \mathbb{N}$

Limit of a Sequence (Definition)

Definition

A sequence $\{a_n\}$ has the **limit** L, denoted $\lim_{n\to\infty} a_n = L$, if successive terms approach L as n increases without bound.

If $\lim_{n\to\infty} a_n = L$, then the sequence **converges** to L.

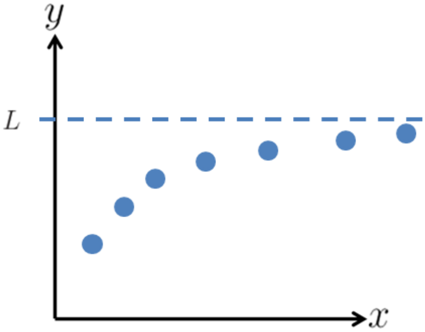
If $\lim_{n\to\infty} a_n = \infty$, then the sequence **diverges** to ∞.

If $\lim_{n\to\infty} a_n = -\infty$, then the sequence **diverges** to $-\infty$.

If $\lim_{n\to\infty} a_n = \text{DNE}$, then the sequence **diverges by oscillation**.

DNE $\equiv$ "**D**oes **N**ot **E**xist"

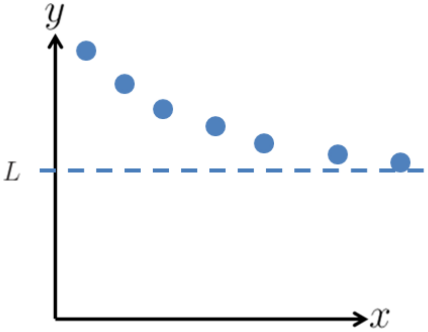
Convergent Sequence (Plot)



Convergent Sequence

$$\lim_{n\to\infty} a_n = L$$

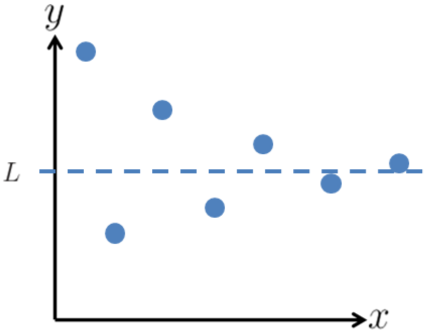
Convergent Sequence (Plot)



Convergent Sequence

$$\lim_{n \to \infty} a_n = L$$

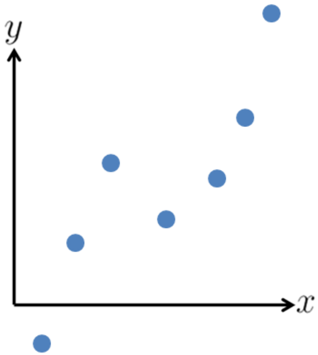
Convergent Sequence (Plot)



Convergent Sequence

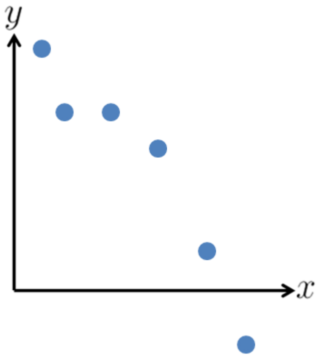
$$\lim_{n\to\infty} a_n = L$$

Divergent Sequence (Plot)



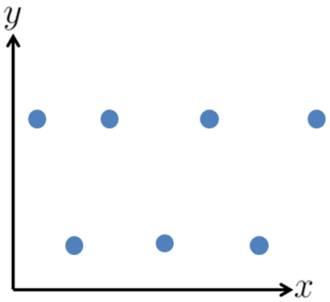
Divergence to ∞

Divergent Sequence (Plot)



Divergence to $-\infty$

Divergent Sequence (Plot)



Divergence by Oscillation

Generating Curves

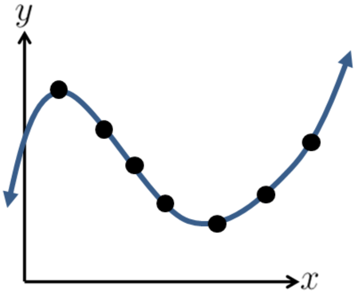
Definition

Let $\{a_n\}$ be a sequence.

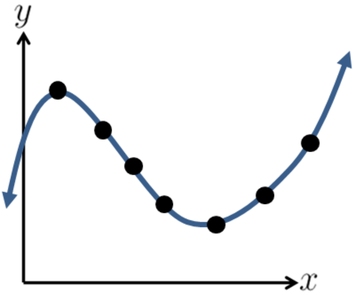
Then function f is a **generating curve** for sequence $\{a_n\}$ if

$$f \in C[1,\infty) \text{ s.t. } a_n = f(n) \quad \forall n \in \mathbb{N}$$

i.e. A generating curve contains every point of the sequence.



Generating Curve Theorem (GCT)



Theorem

(Generating Curve Theorem)

Given sequence $\{a_n\}_{n=1}^{\infty}$ *and function* $f \in C[1,\infty)$ *s.t.* $a_n = f(n) \ \ \forall n \in \mathbb{N}$.

Then, $\lim_{x\to\infty} f(x) = L \implies \lim_{n\to\infty} a_n = L$.

Generating Curve Theorem (GCT)

Theorem

(Generating Curve Theorem)

Given sequence $\{a_n\}_{n=1}^{\infty}$ *and function* $f \in C[1,\infty)$ *s.t.* $a_n = f(n) \;\; \forall n \in \mathbb{N}$.

Then, $\lim\limits_{x\to\infty} f(x) = L \implies \lim\limits_{n\to\infty} a_n = L$.

WARNING: The **converse** to the GCT is not true in general!

i.e. $\lim\limits_{n\to\infty} a_n = L \;\not\Longrightarrow\; \lim\limits_{x\to\infty} f(x) = L$ in general.

Here is a **counterexample**:

Let $a_n = \cos(2\pi n)$ and $f(x) = \cos(2\pi x)$.

Then $\lim\limits_{n\to\infty} a_n = 1$, but $\lim\limits_{x\to\infty} f(x) =$ DNE

WARNING: The generating curve for sequence $a_n = n!$ is the **Gamma Function** $f(x) = \Gamma(x+1)$, which is far too complicated to work with!

The GCT works best when the sequence contains no trig fcn's & no factorials.

Generating Curve Theorem (GCT)

Theorem

(Generating Curve Theorem)

Given sequence $\{a_n\}_{n=1}^{\infty}$ *and function* $f \in C[1,\infty)$ *s.t.* $a_n = f(n) \ \ \forall n \in \mathbb{N}$.

Then, $\lim_{x\to\infty} f(x) = L \implies \lim_{n\to\infty} a_n = L$.

PROOF: Take **Advanced Calculus**.

Limit of a Sequence (Properties)

Theorem

- $\lim_{n\to\infty} k = k$ *(where $k \in \mathbb{R}$)*
- $\lim_{n\to\infty} ka_n = k \lim_{n\to\infty} a_n$
- $\lim_{n\to\infty} (a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$
- $\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$
- $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, *provided* $\lim_{n\to\infty} b_n \neq 0$
- $\lim_{n\to\infty} a_n^p = \left[\lim_{n\to\infty} a_n\right]^p$, *provided* $p > 0$ *and* $a_n \geq 0$.

Limit of a Sequence (Properties)

Theorem

- $\lim_{n\to\infty} k = k$ *(where $k \in \mathbb{R}$)*
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PROOF: Take **Advanced Calculus.**

Limit of a Sequence (Theorems)

Theorem

(Continous Function Theorem for Sequences)

Given sequence $\{a_n\}_{n=1}^{\infty}$ *s.t.* $\lim_{n\to\infty} a_n = L \in \mathbb{R}$ *and* ***continuous*** *function* $f(x)$.

Then, $\lim_{n\to\infty} f(a_n) = f\left(\lim_{n\to\infty} a_n\right) = f(L)$.

Theorem

(Squeeze Theorem for Sequences)

Given $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$ *s.t.* $a_n \leq b_n \leq c_n \;\; \forall n \geq N$ *for some* N.

Then, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L \in \mathbb{R} \implies \lim_{n\to\infty} b_n = L$

Squeeze Theorem is useful when a sequence involves a **factorial**.

Limit of a Sequence (Theorems)

Theorem

(Continous Function Theorem for Sequences)

Given sequence $\{a_n\}_{n=1}^{\infty}$ *s.t.* $\lim_{n\to\infty} a_n = L \in \mathbb{R}$ *and* ***continuous*** *function* $f(x)$.

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PROOF: Take **Advanced Calculus**.

Theorem

(Squeeze Theorem for Sequences)

Given $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=1}^{\infty}$ *s.t.* $a_n \leq b_n \leq c_n \ \ \forall n \geq N$ *for some* N.

Then, $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L \in \mathbb{R} \implies \lim_{n\to\infty} b_n = L$

PROOF: Take **Advanced Calculus**.

Monotonicity & Boundedness of Sequences

Definition

Sequence $\{a_n\}$ is **increasing** if $a_1 \leq a_2 \leq a_3 \leq a_4 \leq \cdots$

Sequence $\{a_n\}$ is **decreasing** if $a_1 \geq a_2 \geq a_3 \geq a_4 \geq \cdots$

Sequence $\{a_n\}$ is **monotone** if it's either increasing or decreasing.

Sequence $\{a_n\}$ is **bounded above** if $\exists M \in \mathbb{R}$ s.t. $a_n \leq M \quad \forall n \in \mathbb{N}$

Sequence $\{a_n\}$ is **bounded below** if $\exists m \in \mathbb{R}$ s.t. $a_n \geq m \quad \forall n \in \mathbb{N}$

Sequence $\{a_n\}$ is **bounded** if it's both bounded above & bounded below.

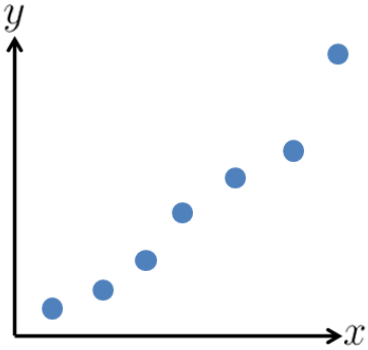
Definition

Sequence $\{a_n\}$ is **eventually increasing** if $\exists N \in \mathbb{N}$ s.t. $a_N \leq a_{N+1} \leq a_{N+2} \leq a_{N+3} \leq \cdots$

Sequence $\{a_n\}$ is **eventually decreasing** if $\exists N \in \mathbb{N}$ s.t. $a_N \geq a_{N+1} \geq a_{N+2} \geq a_{N+3} \geq \cdots$

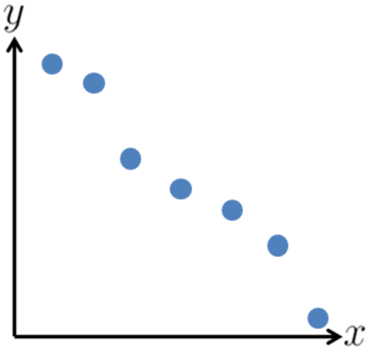
Sequence $\{a_n\}$ is **eventually monotone** if it's either eventually increasing or eventually decreasing.

Monotonicity & Boundedness of Sequences (Plots)



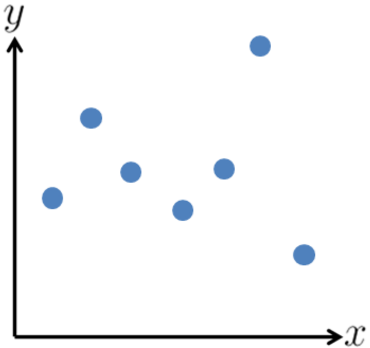
Increasing Sequence

Monotonicity & Boundedness of Sequences (Plots)



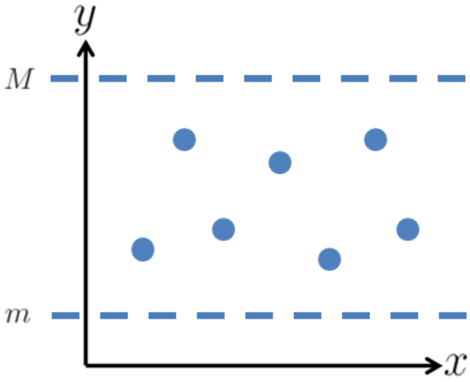
Decreasing Sequence

Monotonicity & Boundedness of Sequences (Plots)



Neither Increasing Nor Decreasing

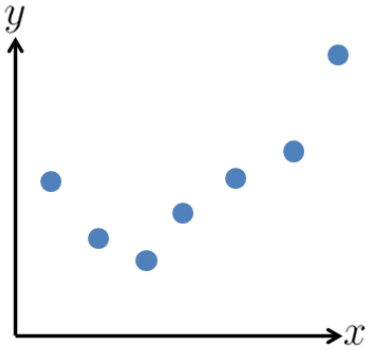
Monotonicity & Boundedness of Sequences (Plots)



Bounded Sequence

$$m \le a_n \le M \quad \forall n \in \mathbb{N}$$

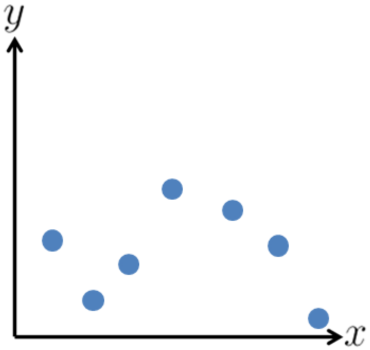
Monotonicity & Boundedness of Sequences (Plots)



Eventually Increasing Sequence

$a_3 \leq a_4 \leq a_5 \leq a_6 \leq a_7 \leq \cdots$

Monotonicity & Boundedness of Sequences (Plots)



Eventually Decreasing Sequence

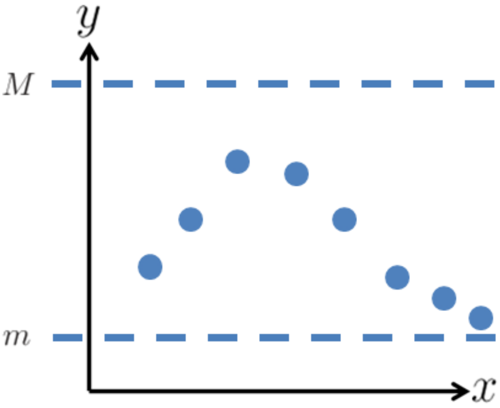
$$a_4 \geq a_5 \geq a_6 \geq a_7 \geq \cdots$$

Bounded Monotone Convergence Theorem (BMCT)

Theorem

(Bounded Monotone Convergence Theorem)

Every bounded eventually monotone sequence converges.



Bounded Monotone Convergence Theorem (BMCT)

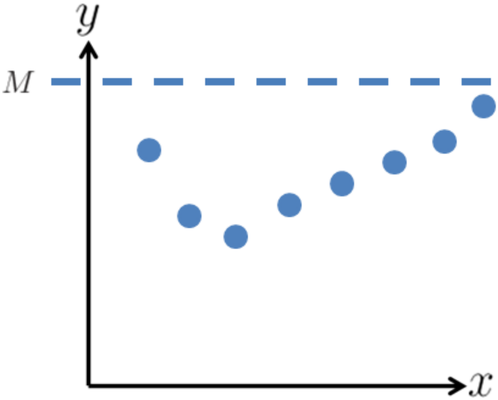
Theorem

(Bounded Monotone Convergence Theorem)

Every bounded eventually monotone sequence converges.

PROOF: Take **Advanced Calculus**.

Corollary to the BMCT

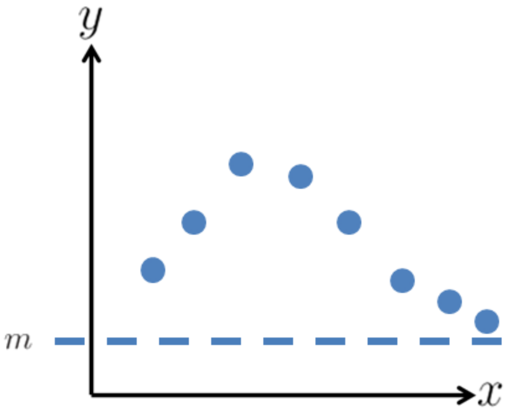


Convergent Sequence

Eventually Increasing

Bounded Above

Corollary to the BMCT



Convergent Sequence
Eventually Decreasing
Bounded Below

Limit of a Geometric Sequence

Definition

A **geometric sequence** has the form $\{ar^n\}$, where $a \neq 0$ and $r \in \mathbb{R}$

Examples of geometric sequences:

- $\{5^n\} = (5, 25, 125, 625, 3125, 15625, \cdots)$
- $\left\{\frac{3}{2^n}\right\} = \left(\frac{3}{2}, \frac{3}{4}, \frac{3}{8}, \frac{3}{16}, \frac{3}{32}, \cdots\right)$
- $\left\{(-1)^n\sqrt{5}\right\} = \left(-\sqrt{5}, \sqrt{5}, -\sqrt{5}, \sqrt{5}, \cdots\right)$
- $\{(-\pi)^n\} = \left(-\pi, \pi^2, -\pi^3, \pi^4, -\pi^5, \pi^6, \cdots\right)$

Theorem

(i) $|r| < 1 \implies \lim_{n\to\infty} r^n = 0$

(ii) $r = 1 \implies \lim_{n\to\infty} r^n = 1$

(iii) $r > 1 \implies \lim_{n\to\infty} r^n = \infty$

(iv) $r \leq -1 \implies \lim_{n\to\infty} r^n = DNE$

Limit of a Geometric Sequence

Theorem

$(i)\ |r| < 1 \implies \lim_{n\to\infty} r^n = 0$ $\qquad (ii)\ r = 1 \implies \lim_{n\to\infty} r^n = 1$

$(iii)\ r > 1 \implies \lim_{n\to\infty} r^n = \infty$ $\qquad (iv)\ r \le -1 \implies \lim_{n\to\infty} r^n = DNE$

PROOF:

(i) CASE I: $r = 0$. Then, $\lim_{n\to\infty} r^n = \lim_{n\to\infty} (0)^n = \lim_{n\to\infty} 0 = 0$

CASE II: $0 < r < 1$.

Then, $r > 0 \implies r^n > 0\ \forall n \in \mathbb{N} \implies \{r^n\}$ is bounded below by zero.
Moreover, $r < 1 \implies r^{n+1} = r^n r < r^n\ \forall n \in \mathbb{N} \implies \{r^n\}$ is decreasing.
Therefore, sequence $\{r^n\}$ is bounded below & decreasing.
$\implies$ sequence $\{r^n\}$ converges by the BMCT $\implies \lim_{n\to\infty} r^n = L$

BWOC, assume $L > 0$. Then, $\lim_{n\to\infty} r^{n+1} = rL < L$ (since $r < 1$)

But, $\lim_{n\to\infty} r^{n+1} = \lim_{n\to\infty} r^n = L \implies L < L \leftarrow$ CONTRADICTION!
Therefore, $L = 0 \implies \lim_{n\to\infty} r^n = 0$

CASE III: $-1 < r < 0$. Then, apply CASE II to the sequence $\{|r|^n\}$.

Limit of a Geometric Sequence

Theorem

$(i)\ |r| < 1 \implies \lim_{n\to\infty} r^n = 0$ $\qquad (ii)\ r = 1 \implies \lim_{n\to\infty} r^n = 1$

$(iii)\ r > 1 \implies \lim_{n\to\infty} r^n = \infty$ $\qquad (iv)\ r \leq -1 \implies \lim_{n\to\infty} r^n = DNE$

PROOF:

(ii) Let $r = 1$. Then, $\lim_{n\to\infty} r^n = \lim_{n\to\infty} (1)^n = \lim_{n\to\infty} 1 = 1$

(iii) Let $r > 1$. Then, $r^{n+1} = r^n r > r^n \implies \{r^n\}$ is increasing.

Moreover, $\{r^n\}$ is unbounded:
BWOC, assume $\{r^n\}$ is bounded.
Then, by defn, $r^n \leq M \quad \forall n \in \mathbb{N}$ for some constant $M > 0$.
But $r^n = M \implies n = \frac{\log M}{\log r}$, and since $\{r^n\}$ is increasing,
for $n > \frac{\log M}{\log r}$, $\qquad r^n > M \leftarrow$ CONTRADICTION!
Therefore, $\{r^n\}$ is increasing & unbounded $\implies \lim_{n\to\infty} r^n = \infty$

(iv) Let $r \leq -1$. Then, $\{r^n\}$ diverges by oscillation $\implies \lim_{n\to\infty} r^n = DNE$

QED

Fin

Fin.