# Geometric Series & Telescoping Series

## Calculus II

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# Finite Sum Rules $(M, N \in \mathbb{N})$

| | |
|---|---|
| Constant Rule | $\sum_{k=1}^{N} c = Nc$ |
| Constant Multiple Rule | $\sum_{k=1}^{N} ca_k = c \sum_{k=1}^{N} a_k$ |
| Sum/Diff Rule | $\sum_{k=1}^{N} (a_k \pm b_k) = \sum_{k=1}^{N} a_k \pm \sum_{k=1}^{N} b_k$ |
| Lump-Sum Rule | $1 < M < N \implies \sum_{k=1}^{M} a_k + \sum_{k=M+1}^{N} a_k = \sum_{k=1}^{N} a_k$ |
| Sum of Integers | $\sum_{k=1}^{N} k = \frac{N(N+1)}{2}$ |
| Sum of Squares | $\sum_{k=1}^{N} k^2 = \frac{N(N+1)(2N+1)}{6}$ |
| Sum of Cubes | $\sum_{k=1}^{N} k^3 = \frac{N^2(N+1)^2}{4}$ |

# Infinite Series (Definition)

## Definition

Let $\{a_n\}$ be a sequence. Then the $n^{th}$ **partial sum** is defined by $S_n := \sum_{k=1}^{n} a_k$

## Definition

An **infinite series**

$$\underbrace{\sum_{k=1}^{\infty} a_k}_{closed\ form} = \underbrace{a_1 + a_2 + a_3 + a_4 + a_5 + \cdots}_{expansion}$$

($k$ is the **index**)

is a sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ where $S_n := \sum_{k=1}^{n} a_k$

REMARK: Going forward, "Series" means "Infinite Series".

# Infinite Series (Convergence)

Going forward, if the starting point of a series does not matter,

$$\text{the series } \sum_{k=N}^{\infty} a_k \text{ will be denoted as } \sum a_k.$$

## Definition

Series $\sum a_k$ **converges** $\iff$ Sequence of Partial Sums $\{S_n\}$ **converges**.

Series $\sum a_k$ **diverges** $\iff$ Sequence of Partial Sums $\{S_n\}$ **diverges**.

Using this definition to test a series for convergence is often too tedious.
Many useful convergence tests will be developed throughout this chapter.

## Definition

Let series $\sum a_k$ converge with partial sum sequence $\{S_n\}$.

Then its **sum** is $\sum a_k = \lim_{n\to\infty} S_n$

# Infinite Series (Linearity)

## Theorem

*(Series Linearity Theorem)*

$(i)$ *Series* $\sum a_k$ *and* $\sum b_k$ *both converge* $\implies \sum(a_k + b_k)$ *converges.*

$(ii)$ *Series* $\sum a_k$ *and* $\sum b_k$ *both converge* $\implies \sum(a_k - b_k)$ *converges.*

$(iii)$ *Series* $\sum a_k$ *converges* $\implies \sum ca_k$ *converges, where* $c \in \mathbb{R}$.

# Infinite Series (Linearity)

## Theorem

$(i)$ *Series* $\sum a_k$ *and* $\sum b_k$ *both converge* $\implies$ $\sum(a_k + b_k)$ *converges.*

$(ii)$ *Series* $\sum a_k$ *and* $\sum b_k$ *both converge* $\implies$ $\sum(a_k - b_k)$ *converges.*

$(iii)$ *Series* $\sum a_k$ *converges* $\implies$ $\sum c a_k$ *converges, where* $c \in \mathbb{R}$*.*

<u>PROOF:</u>

Let series $\sum a_k, \sum b_k$ both converge.
Let $\{A_n\}, \{B_n\}$ be the sequences of partial sums for $\sum a_k, \sum b_k$ respectively.
Then $\{A_n\}, \{B_n\}$ converge to finite values $L_A, L_B$ respectively.

$$\implies \lim_{n\to\infty} A_n = L_A, \qquad \lim_{n\to\infty} B_n = L_B$$

$$\implies \lim_{n\to\infty}(A_n + B_n) = L_A + L_B, \quad \lim_{n\to\infty}(A_n - B_n) = L_A - L_B, \quad \lim_{n\to\infty} cA_n = cL_A$$

$\implies L_A + L_B, \quad L_A - L_B, \quad cL_A$ are all finite values.

$\implies \{A_n + B_n\}, \{A_n - B_n\}, \{cA_n\}$ are all convergent sequences.

$\implies \sum(a_k + b_k), \sum(a_k - b_k), \sum c a_k$ are all convergent series.

QED

### Theorem

*(Divergent Sum Theorem)*

$\sum a_k$ *converges and* $\sum b_k$ *diverges* $\implies \sum (a_k + b_k)$ *diverges.*

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PROOF:

Let $\sum a_k$ converge and $\sum b_k$ diverge.

BWOC, assume $\sum(a_k + b_k)$ converges.

Then, by the Series Linearity Theorem, $\sum[(a_k + b_k) - a_k]$ converges.

$\implies \sum b_k$ converges $\leftarrow$ CONTRADICTION!

Therefore, $\sum(a_k + b_k)$ must diverge.

QED

BWOC $\equiv$ "**B**y **W**ay **O**f **C**ontradiction"

# Infinite Series (Linearity & Divergence)

**Theorem**

*(Divergent Difference Theorem)*

$\sum a_k$ *converges and* $\sum b_k$ *diverges* $\implies \sum(a_k - b_k)$ *diverges.*

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*(Divergent Difference Theorem)*

$\sum a_k$ *converges and* $\sum b_k$ *diverges* $\implies$ $\sum(a_k - b_k)$ *diverges.*

PROOF:

Let $\sum a_k$ converge and $\sum b_k$ diverge.

BWOC, assume $\sum(a_k - b_k)$ converges.

Then, by the Series Linearity Theorem, $\sum [a_k - (a_k - b_k)]$ converges.

$\implies$ $\sum b_k$ converges $\leftarrow$ CONTRADICTION!

Therefore, $\sum(a_k - b_k)$ must diverge.

QED

BWOC $\equiv$ "**B**y **W**ay **O**f **C**ontradiction"

# Infinite Series (Linearity & Divergence)

**Theorem**

*(Divergent Multiple Theorem)*

$\sum a_k$ *diverges* $\implies \sum ca_k$ *diverges, where* $c \neq 0$

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*(Divergent Multiple Theorem)*

$\sum a_k$ *diverges* $\implies \sum ca_k$ *diverges, where* $c \neq 0$

<u>PROOF:</u>

Let $\sum a_k$ diverge and $c \neq 0$.

BWOC, assume $\sum ca_k$ converges.

Then, by the Series Linearity Theorem, $\sum \left[\frac{1}{c}(ca_k)\right]$ converges.

$\implies \sum a_k$ converges $\leftarrow$ CONTRADICTION!

Therefore, $\sum ca_k$ must diverge.

QED

BWOC $\equiv$ "**B**y **W**ay **O**f **C**ontradiction"

# Arithmetic Series

## Definition

An **arithmetic series** has the form $\sum(ak+b)$

Examples of arithmetic series:

$$\sum_{k=1}^{\infty} k = 1+2+3+4+5+\cdots$$

$$\sum_{k=1}^{\infty} 2k = 2+4+6+8+10+\cdots$$

$$\sum_{k=-1}^{\infty} (1-k) = 2+1+0-1-2-\cdots$$

## Theorem

*(Arithmetic Series Theorem)*

*The* ***arithmetic series*** $\sum_{k=1}^{\infty}(ak+b)$ *converges* $\iff a=b=0$

# Arithmetic Series

## Theorem

*The **arithmetic series** $\sum_{k=1}^{\infty}(ak+b)$ converges* $\iff a=b=0$

PROOF:

($\Leftarrow$) Suppose $a=b=0$. Then,

$$\sum_{k=1}^{\infty}(ak+b)=\sum_{k=1}^{\infty}0=0+0+0+0+\cdots=0 \implies \sum_{k=1}^{\infty}(ak+b) \text{ converges.}$$

($\Rightarrow$) Suppose $\sum_{k=1}^{\infty}(ak+b)$ converges.

Then, the partial sum sequence $\{S_n\}$ converges, where

$$S_n=\sum_{k=1}^{n}(ak+b)=a\sum_{k=1}^{n}k+\sum_{k=1}^{n}b=\frac{an(n+1)}{2}+nb \qquad \text{(Finite Sum Rules)}$$

But $\lim_{n\to\infty} n=\lim_{n\to\infty}\frac{n(n+1)}{2}=\infty$, so $\{S_n\}$ converges only if $a=b=0$.

QED

# Geometric Series

## Definition

A **geometric series** has the form $\sum ar^k$, where $a \neq 0$ and $r \in \mathbb{R}$.

Examples of geometric series:

$$\sum_{k=0}^{\infty} 2^k = 1 + 2 + 4 + 8 + 16 + \cdots$$

$$\sum_{k=0}^{\infty} (-2)^k = 1 - 2 + 4 - 8 + 16 - \cdots$$

$$\sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$$

## Theorem

*(Geometric Series Theorem)*

$|r| < 1 \implies$ ***geometric series*** $\sum_{k=0}^{\infty} ar^k$ *converges and* $\sum_{k=0}^{\infty} ar^k = \dfrac{a}{1-r}$

# Geometric Series

## Theorem

*(Geometric Series Theorem)*

$$|r| < 1 \implies \textbf{\textit{geometric series}} \sum_{k=0}^{\infty} ar^k \textit{ converges and } \sum_{k=0}^{\infty} ar^k = \frac{a}{1-r} \quad (a \neq 0)$$

PROOF:

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1} \implies rS_n = ar + ar^2 + ar^3 + \cdots + ar^n$$

$$\implies S_n - rS_n = \left(a + ar + ar^2 + \cdots + ar^{n-1}\right) - \left(ar + ar^2 + ar^3 + \cdots + ar^n\right)$$

$$\implies (1-r)S_n = a - ar^n \implies S_n = \frac{a\left(1-r^n\right)}{1-r}, \quad \text{where} \quad r \neq 1$$

$$|r| < 1 \implies \lim_{n\to\infty} r^n = 0 \implies \lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{a\left(1-r^n\right)}{1-r} = \frac{a(1-0)}{1-r} = \frac{a}{1-r}$$

$$|r| > 1 \implies \lim_{n\to\infty} |r^n| = \infty \implies \lim_{n\to\infty} |S_n| = \infty \implies \sum_{k=0}^{\infty} ar^k \text{ diverges.}$$

$$r = -1 \implies \sum_{k=0}^{\infty} ar^k = \sum_{k=0}^{\infty} a(-1)^k \text{ which diverges} \qquad \text{QED}$$

# Reindexing a Series

Recall the sum of a convergent geometric series:

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

**QUESTION:**

What if the index of a convergent geometric series does not start at zero??

**ANSWER:** **Reindex** the series:

$$\sum_{k=-5}^{\infty} \left(\frac{1}{2}\right)^k = \left(\frac{1}{2}\right)^{-5} + \left(\frac{1}{2}\right)^{-4} + \left(\frac{1}{2}\right)^{-3} + \left(\frac{1}{2}\right)^{-2} + \left(\frac{1}{2}\right)^{-1} + \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k$$

$$\sum_{k=3}^{\infty} \left(\frac{5}{7}\right)^k = -\left(\frac{5}{7}\right)^0 - \left(\frac{5}{7}\right)^1 - \left(\frac{5}{7}\right)^2 + \sum_{k=0}^{\infty} \left(\frac{5}{7}\right)^k$$

REMARK: Reindexing is used in solving certain **differential equations**.

# Telescoping Series

## Definition

A **telescoping series** has internal cancellation of terms in its partial sums.

Basic Forms of Telescoping Series ($f$ is a function):

- $\sum [f(k+1) - f(k)]$
- $\sum [f(k) - f(k+1)]$
- $\sum [f(k) - f(k-1)]$
- $\sum [f(k-1) - f(k)]$

Telescoping Series Toolkit:

- Partial Fraction Decomposition (PFD)
- Properties of Logarithms
- Rationalizing Numerator
- Trig Identities

# Telescoping Series

Given telescoping series $\sum_{k=1}^{\infty}\Big[f(k+1)-f(k)\Big]$, then its $n^{th}$ partial sum is

$$S_n = f(2)-f(1)+f(3)-f(2)+f(4)-f(3)+\cdots+f(n)-f(n-1)+f(n+1)-f(n)$$

$$\implies S_n = f(n+1)-f(1)$$

$$\implies \lim_{n\to\infty} S_n = \lim_{n\to\infty}\Big[f(n+1)-f(1)\Big] = \lim_{n\to\infty}\Big[f(n+1)\Big]-f(1)$$

Now, if sequence $\{S_n\}$ converges, then series $\sum_{k=1}^{\infty}\Big[f(k+1)-f(k)\Big]$ converges

$$\implies \sum_{k=1}^{\infty}\Big[f(k+1)-f(k)\Big] = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\Big[f(n+1)\Big]-f(1)$$

# Fin

Fin.