Positive Series: Integral Test & p-Series

Calculus II

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Bad News about Summing a (Convergent) Series....

Certain types of series can be summed without problem:

- Geometric Series
- Telescoping Series

Unfortunately, in general, series can be hard or impossible to sum.

Case in point:

- Euler showed in 1735 that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{6}$
- To this day, nobody has found the closed-form sum of $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^3}$

Therefore, one can only determine the **convergence** of most series.

To this end, many **convergence tests** will be introduced.

Divergence Test

Theorem

(Divergence Test)

$$\lim_{k\to\infty} a_k \neq 0 \implies \textit{series } \sum a_k \textit{ diverges.}$$

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$$\lim_{k\to\infty} a_k \neq 0 \implies \textit{series } \sum a_k \textit{ diverges.}$$

The **converse** is not true in general: series $\sum a_k$ diverges $\implies \lim_{k\to\infty} a_k \neq 0$

$$\text{COUNTEREXAMPLE: } \sum_{k=1}^{\infty} \frac{1}{k} \text{ diverges even though } \lim_{k\to\infty} \frac{1}{k} = 0$$

The **inverse** is not true in general: $\lim_{k\to\infty} a_k = 0 \implies$ series $\sum a_k$ converges

$$\text{COUNTEREXAMPLE: } \lim_{k\to\infty} \frac{1}{k} = 0 \text{ even though } \sum_{k=1}^{\infty} \frac{1}{k} \text{ diverges}$$

The **contrapositive** is true: series $\sum a_k$ converges $\implies \lim_{k\to\infty} a_k = 0$

Divergence Test

Theorem

(Divergence Test)

$$\lim_{k\to\infty} a_k \neq 0 \implies \textit{series } \sum a_k \textit{ diverges.}$$

<u>PROOF:</u> It's easier to prove the **contrapositive**:

$$\text{series } \sum a_k \text{ converges } \implies \lim_{k\to\infty} a_k = 0$$

Suppose $\sum a_k$ converges. Then $\sum a_k = L$ for some finite value L.

$\implies \lim_{n\to\infty} S_n = L$ where S_n is the n^{th} partial sum of the the series $\sum a_k$

$\implies \lim_{n\to\infty} S_{n-1} = L$ and $S_k - S_{k-1} = a_k$

$\implies \lim_{k\to\infty} a_k = \lim_{k\to\infty} (S_k - S_{k-1}) = \lim_{k\to\infty} S_k - \lim_{k\to\infty} S_{k-1} = L - L = 0$

$\therefore \lim_{k\to\infty} a_k = 0$

QED

Positive Series

Definition

$\sum a_k$ is a **positive series** if each term $a_k \geq 0 \quad \forall k$.

Examples of Positive Series:

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$$

$$\sum_{k=1}^{\infty} \pi^k = \pi + \pi^2 + \pi^3 + \pi^4 + \pi^5 + \cdots$$

$$\sum_{k=0}^{\infty} (-1)^k \cos(k\pi) = 1 + 1 + 1 + 1 + 1 + \cdots$$

Examples that are not Positive Series:

$$\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

$$\sum_{k=0}^{\infty} \cos(k\pi) = 1 - 1 + 1 - 1 + 1 - \cdots$$

Integral Test

Theorem

(Integral Test)

Let function $f(x)$ be continuous & positive for $x \in [N, \infty)$.
Moreover, suppose $a_k = f(k)$ for $k = N, N+1, N+2, N+3, \ldots$ Then:

$$\text{Positive Series } \sum_{k=N}^{\infty} a_k \text{ converges} \iff \text{Integral } \int_N^{\infty} f(x)\,dx \text{ converges}$$

$$\text{Positive Series } \sum_{k=N}^{\infty} a_k \text{ diverges} \iff \text{Integral } \int_N^{\infty} f(x)\,dx \text{ diverges}$$

REMARK: The Integral Test is useless for **factorials** ($k!$) because the generating curve involves the **Gamma Function** $\Gamma(\alpha) := \int_0^{\infty} x^{\alpha-1}e^{-x}\,dx$, and the Gamma Function is too complicated to work with.

REMARK: The only hope for convergence is that f is **eventually decreasing**:

$$\lim_{x\to\infty} f(x) = 0$$

Integral Test

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(Integral Test)

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PROOF:

Recognize that $\sum_{k=N}^{M} a_k$ is a Riemann Sum for $\int_N^M f(x)\,dx$ with $\Delta x_k = 1$.
The details are tedious – see the textbook if interested.

QED

Useful Inequalities for the Integral Test

If the resulting improper integral is hard or impossible to compute, consider applying a useful **inequality** leading to a simpler integral:

- $-1 \leq \cos x \leq 1$ $\quad$ $-1 \leq \sin x \leq 1$ $\quad$ $-\frac{\pi}{2} < \arctan x < \frac{\pi}{2}$
- **For** $x \in \mathbb{R}$: $x^2 \geq 0,\ x^4 \geq 0,\ \cdots,\ x^{2n} \geq 0,\ 2^x > 0,\ e^x > 0,\ x^x \geq 0$
- **For** $x \geq 0$: $\sqrt{x} \geq 0,\ \sqrt[3]{x} \geq 0,\ \sqrt[4]{x} \geq 0,\ \cdots,\ \sqrt[n]{x} \geq 0,\ x^p \geq 0$
- **For** $x \geq 1$: $\sqrt{x} \geq 1,\ \sqrt[3]{x} \geq 1,\ \sqrt[4]{x} \geq 1,\ \cdots,\ \sqrt[n]{x} \geq 1,\ x^p \geq 1$
- **For** $x \geq 1$: $\log_2 x \geq 0,\ \ln x \geq 0,\ \log x \geq 0$
- $A < B \implies -A > -B$ $\quad$ $A > B \implies -A < -B$
  $A \leq B \implies -A \geq -B$ $\quad$ $A \geq B \implies -A \leq -B$
- $A, M, m > 0$ s.t. $M > m \implies AM > Am$ and $\frac{A}{M} < \frac{A}{m}$
- $A, x > 0 \implies A + x > A \implies \frac{1}{A+x} < \frac{1}{A}$
- $A > x > 0 \implies A - x < A \implies \frac{1}{A-x} > \frac{1}{A}$
- f is **positive & increasing** on $[A, B]$ AND $0 < A < B \implies 0 < f(A) < f(B)$
- f is **positive & decreasing** on $[A, B]$ AND $0 < A < B \implies f(A) > f(B) > 0$
- (Integral Dominance Rule) $f(x) \leq g(x) \implies \int_N^\infty f(x)\, dx \leq \int_N^\infty g(x)\, dx$

A Note about Inequality Chains

Every inequality in an **inequality chain** must be pointing in same direction:

$A \le B \le C \le D = E \le F \le G = H$ implies that $A \le H$
$A < B \le C < D = E < F \le G = H$ implies that $A < H$
$A < B < C < D = E < F < G = H$ implies that $A < H$

$A = B \ge C = D \ge E \ge F \ge G \ge H$ implies that $A \ge H$
$A = B \ge C = D \ge E > F \ge G \ge H$ implies that $A > H$
$A = B > C = D > E > F > G > H$ implies that $A > H$

Otherwise, the inequality chain is useless:

$A < B \le C > D = E < F$ implies nothing on how:

- A and D are related
- A and E are related
- A and F are related
- B and D are related
- B and E are related
- B and F are related
- C and F are related

Applying Inequalities for the Integral Test

WORKED EXAMPLE: Test the series $\sum_{k=0}^{\infty} e^{-k^2}$ for convergence.

Observe that for $x \geq 0$: e^{-x^2} is continuous & positive, and

$x^2 \geq x \implies -x^2 \leq -x \implies e^{-x^2} \leq e^{-x}$ (since e^x is **positive** & **increasing**)

Apply the **Integral Test**:

$$\int_0^{\infty} e^{-x^2}\,dx \leq \int_0^{\infty} e^{-x}\,dx = \Big[-e^{-x}\Big]_{x=0}^{x\to\infty} \overset{FTC}{=} \lim_{x\to\infty} \left(-e^{-x}\right) - \left(-e^{-(0)}\right) = 1 < \infty$$

$$\therefore \int_0^{\infty} e^{-x^2}\,dx < \infty \implies \int_0^{\infty} e^{-x^2}\,dx \text{ converges} \implies \boxed{\sum_{k=0}^{\infty} e^{-k^2} \text{ converges}}$$

Applying Inequalities for the Integral Test

WORKED EXAMPLE: Test the series $\displaystyle\sum_{k=1}^{\infty} \frac{\sqrt[3]{k^5+k+1}}{k}$ for convergence.

Observe that for $x \geq 1$: $\dfrac{\sqrt[3]{x^5+x+1}}{x}$ is continuous & positive, and

$x^5+x+1 \geq x^5+x \geq x^5$ and $\sqrt[3]{x}$ is **positive** and **increasing**.

Apply the **Integral Test**:

$$\int_1^{\infty} \frac{\sqrt[3]{x^5+x+1}}{x}\,dx \geq \int_1^{\infty} \frac{\sqrt[3]{x^5}}{x}\,dx = \int_1^{\infty} \frac{x^{5/3}}{x}\,dx \geq \int_1^{\infty} \frac{1}{x}\,dx = \infty$$

$$\therefore \int_1^{\infty} \frac{\sqrt[3]{x^5+x+1}}{x}\,dx \geq \infty \implies \int_1^{\infty} \frac{\sqrt[3]{x^5+x+1}}{x}\,dx \text{ diverges}$$

$$\implies \boxed{\sum_{k=1}^{\infty} \frac{\sqrt[3]{k^5+k+1}}{k} \text{ diverges}}$$

Applying Inequalities for the Integral Test

WORKED EXAMPLE: Test the series $\sum_{k=-2}^{\infty} (7 + 3\sin k)$ for convergence.

Observe that for $x \geq -2$: $7 + 3\sin x$ is continuous & positive, and

$$\sin x \geq -1 \implies 3\sin x \geq -3 \implies 7 + 3\sin x \geq 7 + (-3) = 4$$

Apply the **Integral Test**:

$$\int_{-2}^{\infty} (7 + 3\sin x)\ dx \geq \int_{-2}^{\infty} 4\ dx = \Big[4x\Big]_{x=-2}^{x\to\infty} \overset{FTC}{=} \left(\lim_{x\to\infty} 4x\right) - 4(-2) = \infty + 8 = \infty$$

$$\therefore \int_{-2}^{\infty} (7 + 3\sin x)\ dx \geq \infty \implies \int_{-2}^{\infty} (7 + 3\sin x)\ dx \text{ diverges}$$

$$\implies \boxed{\sum_{k=-2}^{\infty} (7 + 3\sin k) \text{ diverges}}$$

p-Series Test

Theorem

(p-Series Test)

$$p > 1 \implies p\text{-series } \sum_{k=1}^{\infty} \frac{1}{k^p} \text{ converges}$$

$$p \leq 1 \implies p\text{-series } \sum_{k=1}^{\infty} \frac{1}{k^p} \text{ diverges}$$

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PROOF: Apply the Integral Test.

CASE I: $p = 1$.

$$\int_1^\infty \frac{1}{x}\, dx = \Big[\ln x \Big]_{x=1}^{x \to \infty} \overset{FTC}{=} \ln(\infty) - 1 = \infty - 1 = \infty$$

$\therefore$ **Harmonic Series** $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k}$ diverges

p-Series Test

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$$p \leq 1 \implies p\text{-series} \sum_{k=1}^{\infty} \frac{1}{k^p} \text{ diverges}$$

<u>CASE II:</u> $p < 1$.

Then $p < 1 \implies -p > -1 \implies 1 - p > 0$.

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \left[\frac{x^{1-p}}{1-p}\right]_{x=1}^{x\to\infty} \overset{FTC}{=} \frac{(\infty)^{1-p}}{1-p} - \frac{1}{1-p} = \infty - \frac{1}{1-p} = \infty$$

$\therefore$ p-Series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^p}$ diverges

Theorem

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<u>CASE III:</u> $p > 1$.

Then $p > 1 \implies -p < -1 \implies 1 - p < 0$.

$$\int_1^{\infty} \frac{1}{x^p}\,dx = \left[\frac{x^{1-p}}{1-p}\right]_{x=1}^{x\to\infty} \overset{FTC}{=} \frac{(\infty)^{1-p}}{1-p} - \frac{1}{1-p} = 0 - \frac{1}{1-p} = \frac{1}{p-1} < \infty$$

$\therefore$ p-Series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges

QED

Fin.