Positive Series: Comparison Tests

Calculus II

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TTU

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Direct Comparison Test (DCT)

Theorem

(Direct Comparison Test)

$$0 \leq a_k \leq c_k \text{ AND } \sum c_k \text{ converges } \implies \sum a_k \text{ converges}$$

$$b_k \geq d_k \geq 0 \text{ AND } \sum d_k \text{ diverges } \implies \sum b_k \text{ diverges}$$

KEY IDEA: Bound a hard series with a series whose convergence is known.

DISADVANTAGE: Easy to pick the ”wrong” series to compare with, causing the test to fail.

Direct Comparison Test (DCT)

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$$b_k \ge d_k \ge 0 \;\; \textit{AND} \;\; \sum d_k \textit{ diverges} \implies \sum b_k \textit{ diverges}$$

PROOF:

Let $a_k = f(k)$ and $c_k = F(k)$ for $k \ge N_1$ where $N_1 \in \mathbb{N}$.

Then, $0 \le a_k \le c_k$ for $k \ge N_1 \implies 0 \le f(x) \le F(x)$ for $x \ge N_1$

Now, $\sum c_k$ converges $\implies \int_{N_1}^{\infty} F(x)\, dx < \infty$ (by the Integral Test)

$$\implies \int_{N_1}^{\infty} f(x)\, dx \le \int_{N_1}^{\infty} F(x)\, dx < \infty \implies \int_{N_1}^{\infty} f(x)\, dx < \infty$$

Hence, $\sum a_k$ converges (by the Integral Test)

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PROOF:

Let $b_k = G(k)$ and $d_k = g(k)$ for $k \geq N_2$ where $N_2 \in \mathbb{N}$.

Then, $b_k \geq d_k \geq 0$ for $k \geq N_2 \implies G(x) \geq g(x) \geq 0$ for $x \geq N_2$

Now, $\sum d_k$ diverges $\implies \int_{N_2}^{\infty} g(x)\, dx = \infty$ (by the Integral Test)

$$\implies \int_{N_2}^{\infty} G(x)\, dx \geq \int_{N_2}^{\infty} g(x)\, dx = \infty \implies \int_{N_2}^{\infty} G(x)\, dx = \infty$$

Hence, $\sum b_k$ diverges (by the Integral Test)

QED

Useful Inequalities for the Direct Comparison Test

- $-1 \leq \cos x \leq 1 \qquad -1 \leq \sin x \leq 1 \qquad -\frac{\pi}{2} < \arctan x < \frac{\pi}{2}$
- **For** $x \in \mathbb{R}$: $x^2 \geq 0,\ x^4 \geq 0,\ \cdots,\ x^{2n} \geq 0,\ 2^x > 0,\ e^x > 0,\ x^x \geq 0$
- **For** $x \geq 0$: $\sqrt{x} \geq 0,\ \sqrt[3]{x} \geq 0,\ \sqrt[4]{x} \geq 0,\ \cdots,\ \sqrt[n]{x} \geq 0,\ x^p \geq 0$
- **For** $x \geq 1$: $\sqrt{x} \geq 1,\ \sqrt[3]{x} \geq 1,\ \sqrt[4]{x} \geq 1,\ \cdots,\ \sqrt[n]{x} \geq 1,\ x^p \geq 1$
- **For** $x \geq 1$: $\log_2 x \geq 0,\ \ln x \geq 0,\ \log x \geq 0$
- $A < B \implies -A > -B \qquad A > B \implies -A < -B$
  $A \leq B \implies -A \geq -B \qquad A \geq B \implies -A \leq -B$
- $A, M, m > 0$ s.t. $M > m \implies AM > Am$ and $\frac{A}{M} < \frac{A}{m}$
- $A, x > 0 \implies A + x > A \implies \frac{1}{A+x} < \frac{1}{A}$
- $A > x > 0 \implies A - x < A \implies \frac{1}{A-x} > \frac{1}{A}$
- f is **positive & increasing** on $[A, B]$ AND $0 < A < B \implies 0 < f(A) < f(B)$
- f is **positive & decreasing** on $[A, B]$ AND $0 < A < B \implies f(A) > f(B) > 0$

"Tower of Power"

Eventually (i.e. as $x, k \to \infty$):

$\cdots \leq \ln(\ln x) \leq \log_{100} x \leq \ln x \leq \log_2 x \leq \sqrt[100]{x} \leq \sqrt[3]{x} \leq \sqrt{x} \leq x \leq x^2 \leq x^{100} \leq$
$2^x \leq e^x \leq 100^x \leq k! \leq x^x \leq x^{x^x} \leq \cdots$

in other words....

Nested Log's $\leq$ Log's $\leq$ Roots $\leq$ Powers $\leq$ Exp's $\leq$ Factorials $\leq$ Nested Exp's

A Note about Inequality Chains

Every inequality in an **inequality chain** must be pointing in same direction:

$A \le B \le C \le D = E \le F \le G = H$ implies that $A \le H$
$A < B \le C < D = E < F \le G = H$ implies that $A < H$
$A < B < C < D = E < F < G = H$ implies that $A < H$

$A = B \ge C = D \ge E \ge F \ge G \ge H$ implies that $A \ge H$
$A = B \ge C = D \ge E > F \ge G \ge H$ implies that $A > H$
$A = B > C = D > E > F > G > H$ implies that $A > H$

Otherwise, the inequality chain is useless:

$A < B \le C > D = E < F$ implies nothing on how:

A and D are related
A and E are related
A and F are related
B and D are related
B and E are related
B and F are related
C and F are related

Limit Comparison Test (LCT)

Theorem

(Limit Comparison Test)

Let $\sum a_k, \sum b_k, \sum c_k, \sum d_k$ *all be positive series. Then:*

$$\lim_{k\to\infty} \frac{a_k}{c_k} = 0 \text{ AND } \sum c_k \text{ converges } \implies \sum a_k \text{ converges}$$

$$\lim_{k\to\infty} \frac{a_k}{b_k} = L \in (0,\infty) \implies \sum a_k \text{ and } \sum b_k \text{ both converge or both diverge}$$

$$\lim_{k\to\infty} \frac{a_k}{d_k} = \infty \text{ AND } \sum d_k \text{ diverges } \implies \sum a_k \text{ diverges}$$

KEY IDEA: Compare a series with a simpler series that ”looks like” it.

ADVANTAGE: Does not use inequalities & often works when DCT fails.

DISADVANTAGE: Some series are easy with DCT but hard with LCT.

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$$\lim_{k\to\infty} \frac{a_k}{d_k} = \infty \ \textit{ AND } \ \sum d_k \textit{ diverges} \implies \sum a_k \textit{ diverges}$$

PROOF: Take **Advanced Calculus**.

LCT: Simple Series to Compare with

Simple Convergent Series	Simple Divergent Series
Geometric Series $\sum_{k=0}^{\infty} r^k$ with $\|r\| < 1$	Geometric Series $\sum_{k=0}^{\infty} r^k$ with $\|r\| \geq 1$
p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ with $p > 1$	p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ with $p \leq 1$

A Word of Advice

- One can argue that this is the hardest section involving Series Tests.
- The key to picking the right series to compare with is **experience**.
  - Therefore, understand all the examples here, in the book, and in the HW.
  - It's also advised to attempt some of the problems in the book.

Fin.