Positive Series: Ratio Test & Root Test

Calculus II

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TTU

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Ratio Test

Theorem

(Ratio Test)

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} < 1 \implies \text{positive series } \sum a_k \text{ converges}$$

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} > 1 \implies \text{positive series } \sum a_k \text{ diverges}$$

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \infty \implies \text{positive series } \sum a_k \text{ diverges}$$

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = 1 \text{ or DNE} \implies \text{Ratio Test fails}$$

Works nicely with:

- **factorials** (e.g. $k!$)
- **simple powers** (e.g. k^3)
- **simple exponentials** (e.g. 2^k)
- **product chains** (e.g. $1 \cdot 3 \cdot 5 \cdots (2k-1)$).

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- Useless with most **trig fcns** b/c: $\lim_{k\to\infty} \sin k = \text{DNE}$ $\qquad \lim_{k\to\infty} \cos k = \text{DNE}$
- Tedious to use with most **rational functions of polynomials**.

  e.g. $a_k = \dfrac{k^3 - 2k^2 + k + 1}{k^4 + k^3 + k^2 + 2}$

Ratio Test

Theorem

(Ratio Test)

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} < 1 \implies \textit{positive series } \sum a_k \textit{ converges}$$

PROOF:

See the textbook for the case when $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} < 1$.

Ratio Test

Theorem

(Ratio Test)

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} > 1 \implies \textit{positive series } \sum a_k \textit{ diverges}$$

PROOF:

Suppose $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} > 1$. Then

Sequence $\{a_n\}$ is positive, unbounded, and eventually increasing

$\implies$ Partial Sum Sequence $\{S_n\}$ is positive, unbounded, and increasing

$\implies \lim_{n\to\infty} S_n = \infty$

$\implies$ Positive Series $\sum a_k$ diverges.

Ratio Test

Theorem

(Ratio Test)

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \infty \implies \textit{positive series } \sum a_k \textit{ diverges}$$

PROOF:

Suppose $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \infty$. Then

Sequence $\{a_n\}$ is positive, unbounded, and eventually increasing

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Ratio Test

Theorem

(Ratio Test)

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = 1 \text{ or DNE} \implies \text{Ratio Test fails}$$

PROOF:

For the case that $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = 1$, it suffices to observe that

$$\sum_{k=1}^{\infty} \frac{1}{k} \text{ diverges yet } \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\frac{1}{k+1}}{\frac{1}{k}} = \lim_{k\to\infty} \frac{k}{k+1} = 1$$

$$\sum_{k=1}^{\infty} \frac{1}{k^2} \text{ converges yet } \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\frac{1}{(k+1)^2}}{\frac{1}{k^2}} = \lim_{k\to\infty} \frac{k^2}{(k+1)^2} = 1$$

Therefore, the Ratio Test is inconclusive when $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = 1$

Ratio Test

Theorem

(Ratio Test)

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = 1 \textit{ or DNE} \implies \textit{Ratio Test fails}$$

PROOF:

For the case that $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} =$ DNE, it suffices to observe that

$\sum_{k=1}^{\infty}(2+\sin k)$ diverges yet $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{2+\sin(k+1)}{2+\sin k} =$ DNE

$\sum_{k=1}^{\infty} \frac{|\sin k|}{k^2}$ converges yet $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\frac{|\sin(k+1)|}{(k+1)^2}}{\frac{|\sin k|}{k^2}} = \lim_{k\to\infty} \frac{|\sin(k+1)|}{|\sin k|} =$ DNE

Therefore, the Ratio Test is inconclusive when $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} =$ DNE

QED

Ratio Test (Examples)

WORKED EXAMPLE: Test the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k!}$ for convergence.

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{\frac{1}{(k+1)!}}{\frac{1}{k!}} = \lim_{k\to\infty} \frac{k!}{(k+1)!} = \lim_{k\to\infty} \frac{k!}{(k+1)k!} = \lim_{k\to\infty} \frac{1}{k+1} = 0 < 1$$

Therefore, by the Ratio Test, series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k!}$ converges

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WORKED EXAMPLE: Test the series $\displaystyle\sum_{k=-9}^{\infty} k!$ for convergence.

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{(k+1)!}{k!} = \lim_{k\to\infty} \frac{(k+1)k!}{k!} = \lim_{k\to\infty} (k+1) = \infty$$

Therefore, by the Ratio Test, series $\displaystyle\sum_{k=-9}^{\infty} k!$ diverges

Root Test

Theorem

$$\lim_{k\to\infty} \sqrt[k]{a_k} < 1 \implies \textit{positive series } \sum a_k \textit{ converges}$$

$$\lim_{k\to\infty} \sqrt[k]{a_k} > 1 \implies \textit{positive series } \sum a_k \textit{ diverges}$$

$$\lim_{k\to\infty} \sqrt[k]{a_k} = \infty \implies \textit{positive series } \sum a_k \textit{ diverges}$$

$$\lim_{k\to\infty} \sqrt[k]{a_k} = 1 \textit{ or DNE} \implies \textit{Root Test fails}$$

Works nicely with:

- **heavy powers of k** $\left(\text{e.g. } [f(k)]^k\text{, where } f \text{ is a } \textbf{continuous fcn}\right)$
- **heavy exponentials** $\left(\text{e.g. } 9^{k^2}, 7^{\ln k}, 3^{\sin k}, \ldots\right)$.

Useless with:

- **factorials** (e.g. $k!$).

Root Test

Theorem

$$\lim_{k\to\infty} \sqrt[k]{a_k} < 1 \implies \textit{positive series } \sum a_k \textit{ converges}$$

$$\lim_{k\to\infty} \sqrt[k]{a_k} > 1 \implies \textit{positive series } \sum a_k \textit{ diverges}$$

$$\lim_{k\to\infty} \sqrt[k]{a_k} = \infty \implies \textit{positive series } \sum a_k \textit{ diverges}$$

$$\lim_{k\to\infty} \sqrt[k]{a_k} = 1 \textit{ or DNE} \implies \textit{Root Test fails}$$

PROOF:

Take **Advanced Calculus**.

Root Test (Examples)

WORKED EXAMPLE: Test the series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(k+2)^k}$ for convergence.

$$\lim_{k\to\infty} \sqrt[k]{a_k} = \lim_{k\to\infty} \sqrt[k]{\frac{1}{(k+2)^k}} = \lim_{k\to\infty} \left[\frac{1}{(k+2)^k}\right]^{1/k} = \lim_{k\to\infty} \frac{1}{k+2} = 0 < 1$$

Therefore, by the Root Test, series $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(k+2)^k}$ converges

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WORKED EXAMPLE: Test the series $\displaystyle\sum_{k=6}^{\infty} \pi^{-k/2} k^{5k}$ for convergence.

$$\lim_{k\to\infty} \sqrt[k]{a_k} = \lim_{k\to\infty} \left[\pi^{-k/2} k^{5k}\right]^{1/k} = \lim_{k\to\infty} \pi^{-1/2} k^5 = \lim_{k\to\infty} \frac{k^5}{\sqrt{\pi}} = \infty > 1$$

Therefore, by the Root Test, series $\displaystyle\sum_{k=6}^{\infty} \pi^{-k/2} k^{5k}$ diverges

Relationship between the Ratio Test & Root Test

Typically, the Ratio Test & Root Test either both succeed or both fail.

However, there are series where Ratio Test fails but Root Test succeeds:

WORKED EXAMPLE: Test the series $\displaystyle\sum_{k=0}^{\infty} \frac{1}{2^{k+(-1)^k}}$ for convergence.

Expand the series: $\displaystyle\sum_{k=0}^{\infty} \frac{1}{2^{k+(-1)^k}} = \frac{1}{2} + 1 + \frac{1}{8} + \frac{1}{4} + \frac{1}{32} + \frac{1}{16} + \frac{1}{128} + \frac{1}{64} + \cdots$

Then, $\displaystyle\frac{a_{k+1}}{a_k} = \begin{cases} 2 & \text{, if } k \text{ is even} \\ 1/8 & \text{, if } k \text{ is odd} \end{cases} \implies \left\{\frac{a_{k+1}}{a_k}\right\}_{k=0}^{\infty} = \left(2, \frac{1}{8}, 2, \frac{1}{8}, 2, \frac{1}{8}, \cdots\right)$

$\displaystyle\implies \left\{\frac{a_{k+1}}{a_k}\right\}_{k=0}^{\infty}$ oscillates $\displaystyle\implies \lim_{k\to\infty} \frac{a_{k+1}}{a_k} =$ DNE $\implies$ Ratio Test fails

However,

$$\lim_{k\to\infty} \sqrt[k]{a_k} = \lim_{k\to\infty} \frac{1}{2^{[k+(-1)^k]/k}} = \lim_{k\to\infty} \frac{1}{2^{1+(-1)^k/k}} = \frac{1}{2} \lim_{k\to\infty} \frac{1}{2^{(-1)^k/k}} = \frac{1}{2} < 1$$

$\therefore$ by the Root Test, Series Converges

Be Careful with Factorial Expressions

Definition

$$k! := k(k-1)(k-2)\cdots(3)(2)(1) \qquad\qquad 0! := 1$$

CAUTION:

- $(k+3)! \neq k! + 3, \qquad (k+3)! \neq k! + 3!$
  - Rather, $(k+3)! = (k+3)(k+2)(k+1)k!$
- $(3k)! \neq 3k!, \qquad (3k)! \neq 3!k!$
  - Rather, $(3k)! = (3k)(3k-1)(3k-2)(3k-3)\cdots(k+2)(k+1)k!$
- $k!k! \neq \left(k^2\right)!$
  - Rather, $k!k! = (k!)^2$

Fin.