Alternating Series

Calculus II

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TTU

02 April 2014

Alternating Series

Definition

An **alternating series** has one of the following forms:

$$\sum(-1)^k a_k,\ \sum(-1)^{k+1} a_k \text{ or } \sum(-1)^{k-1} a_k, \text{ where } a_k \geq 0 \quad \forall k$$

In other words, alternating series exhibit one of the two **sign patterns**:

$+ - + - + - + - + - + - \cdots$ OR $- + - + - + - + - + - + \cdots$

Series	Expansion	Sign Pattern
$\sum_{k=0}^{\infty} a_k$	$a_0 + a_1 + a_2 + a_3 + a_4 + a_5 + \cdots$	$+ + + + + + + + + \cdots$
$\sum_{k=0}^{\infty}(-1)^{k-1} a_k$	$-a_0 + a_1 - a_2 + a_3 - a_4 + a_5 - \cdots$	$- + - + - + - + \cdots$
$\sum_{k=0}^{\infty}(-1)^{k} a_k$	$a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + \cdots$	$+ - + - + - + - \cdots$
$\sum_{k=0}^{\infty}(-1)^{k+1} a_k$	$-a_0 + a_1 - a_2 + a_3 - a_4 + a_5 - \cdots$	$- + - + - + - + \cdots$
$\sum_{k=0}^{\infty}(-1)^{k(k+1)/2} a_k$	$a_0 - a_1 - a_2 + a_3 + a_4 - a_5 - \cdots$	$+ - - + + - - + \cdots$

The green series above is a **positive series**.
The blue series above are **alternating series**.
The red series above is **neither positive nor alternating**.

Not all Powers of -1 Result in Alternating Series

- Some powers of -1 result in a **positive series**:
  - $\sum(-1)^{2k+2}$ $\left[\text{Since } (-1)^{2k+2} = (-1)^{2(k+1)} = (-1)^{(\text{even \#})} = 1\right]$
  - $\sum 2^{-k-(-1)^k}$ $\left[\text{Since } 2^x > 0\right]$
- Others result in a series that's **neither positive nor alternating**:

Series	**Sign Pattern**
$\sum_{k=0}^{\infty}(-1)^{k(k+1)/2}$	$+ - - + + - - + + - - + + - - + + - - + + - -$
$\sum_{k=0}^{\infty}(-1)^{\lfloor k/2 \rfloor}$	$+ + - - + + - - + + - - + + - - + + - - + + -$
$\sum_{k=0}^{\infty}(-1)^{\lceil k/3 \rceil}$	$+ - - - + + + - - - + + + - - - + + + - - - +$

$\lfloor x \rfloor$ is the **floor function**: $\lfloor 3 \rfloor = 3, \lfloor 3.1 \rfloor = 3, \lfloor 3.9 \rfloor = 3$ (Round down)

$\lceil x \rceil$ is the **ceiling function**: $\lceil 3 \rceil = 3, \lceil 3.1 \rceil = 4, \lceil 3.9 \rceil = 4$ (Round up)

Alternating Series Test

Theorem

(Alternating Series Test)

Let $\{a_k\}$ *be a positive, eventually decreasing sequence s.t.* $\lim_{k\to\infty} a_k = 0$. *Then*

Alternating Series $\sum(-1)^k a_k, \sum(-1)^{k+1} a_k, \sum(-1)^{k-1} a_k$ *all converge*

If $\lim_{k\to\infty} a_k \neq 0$, then the alternating series diverges (by the Divergence Test)

Absolute & Conditional Convergence

Definition

Let $\sum a_k$ be any series. Then

$$\sum a_k \textbf{ converges absolutely} \iff \text{Positive Series } \sum |a_k| \text{ converges}$$

$$\sum a_k \textbf{ converges conditionally} \iff \sum a_k \text{ converges, but } \sum |a_k| \text{ diverges}$$

Theorem

(Absolute Convergence Test)

$$\textit{Positive Series } \sum |a_k| \textit{ converges} \implies \textit{Series } \sum a_k \textit{ converges}$$

- The Absolute Convergence Test is the only directly applicable test for series that are **neither positive nor alternating**.
- Series that are **neither positive nor alternating and fails the Absolute Convergence Test** will not be considered.
  - Such series requires tests (Dirichlet's Test, Abel's Test, Ermakoff's Test) seen in **Advanced Calculus**.

WORKED EXAMPLE: Test the series $\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$ for absolute convergence.

The series is of the form $\sum (-1)^k a_k$, where $a_k = \dfrac{1}{k!}$, so it's alternating.

$\displaystyle\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{k!} = 0$ and sequence $\{a_k\} = \left\{\frac{1}{k!}\right\}_{k=0}^{\infty}$ is decreasing.

$\therefore$ By the Alternating Series Test, $\boxed{\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \text{ converges}}$

Moreover, $\displaystyle\sum_{k=0}^{\infty} |a_k| = \sum_{k=0}^{\infty} \left|\frac{(-1)^k}{k!}\right| = \sum_{k=0}^{\infty} \frac{1}{k!}$ converges (by the Ratio Test)

$\therefore$ $\boxed{\displaystyle\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \text{ absolutely converges}}$

Absolute & Conditional Convergence

WORKED EXAMPLE: Test series $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$ for absolute convergence.

The series is of the form $\sum(-1)^{k+1}a_k$, where $a_k = \dfrac{1}{\sqrt{k}}$, so it's alternating.

$\displaystyle\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{1}{\sqrt{k}} = 0$ and sequence $\{a_k\} = \left\{\dfrac{1}{\sqrt{k}}\right\}_{k=1}^{\infty}$ is decreasing.

$\therefore$ By the Alternating Series Test, $\boxed{\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}} \text{ converges}}$

Moreover, $\displaystyle\sum_{k=1}^{\infty} |a_k| = \sum_{k=1}^{\infty} \left|\frac{(-1)^k}{\sqrt{k}}\right| = \sum_{k=1}^{\infty} \frac{1}{k^{1/2}}$ which is a divergent p-series.

$\therefore$ $\boxed{\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}} \text{ conditionally converges}}$

Rearrangment of Series

- Addition is **commutative**: $3+5=5+3$
- Addition is **associative**: $3+(5+7)=(3+5)+7$
- So, for **finite series**, the terms can be **rearranged**:

  e.g. $\displaystyle\sum_{k=1}^{3} k = 1+2+3 = 1+3+2 = 2+3+1 = 3+2+1 = 6$
- An **absolutely convergent series** also can be **rearranged**:

$$\text{e.g. } \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2} = -1+\frac{1}{4}-\frac{1}{9}+\frac{1}{16}-\frac{1}{25}+\frac{1}{36}-\frac{1}{49}+\cdots$$

$$= \left(-1+\frac{1}{4}\right)+\left(-\frac{1}{9}+\frac{1}{16}\right)+\left(-\frac{1}{25}+\frac{1}{36}\right)+\cdots$$

$$= \left(-1-\frac{1}{9}\right)+\left(\frac{1}{4}+\frac{1}{16}\right)+\left(-\frac{1}{25}-\frac{1}{49}\right)+\cdots$$

Rearrangment of Series

- **Rearranging** a **conditionally convergent series** yields **different sums**:

  e.g. $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \ln 2,$

  but $\displaystyle\left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \cdots = \frac{1}{2}\ln 2$

- **Rearranging** a **divergent series** yields **downright weird results**:

  e.g. $\displaystyle\sum_{k=1}^{\infty}(-1)^k = -1 + 1 - 1 + 1 - 1 + \cdots$ diverges by oscillation,

  but $(-1+1) + (-1+1) + (-1+1) + (-1+1) + \cdots = 0$

  and $(1+1-1) + (1+1-1) + (1+1-1) + \cdots = +\infty$

  and $(1-1-1) + (1-1-1) + (1-1-1) + \cdots = -\infty$

Fin

Fin.