Power Series

Calculus II

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Power Series (Definition)

Definition

A **power series** has the form

$$\sum_{k=0}^{\infty} a_k (x-c)^k = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

- A power series with $c = 0$ simplifies to:
  $\sum_{k=0}^{\infty} a_k x^k = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$
- A **polynomial** is a **finite power series** :
  e.g. $x^3 - 2x^2 + 3x - 4 = \sum_{k=0}^{\infty} a_k x^k$ with $\begin{cases} a_0 = -4, a_1 = 3, a_2 = -2, a_3 = 1 \\ a_4 = a_5 = a_6 = a_7 = \cdots = 0 \end{cases}$

Power Series (Convergence)

Proposition

*Given a **power series** $\sum_{k=0}^{\infty} a_k(x-c)^k$, exactly one of the following is true:*

$\sum a_k(x-c)^k$ *converges for all* x
$\sum a_k(x-c)^k$ *converges only for* $x=c$
$\sum a_k(x-c)^k$ *converges* $\forall x \in (c-R, c+R)$
$\sum a_k(x-c)^k$ *converges* $\forall x \in [c-R, c+R)$
$\sum a_k(x-c)^k$ *converges* $\forall x \in (c-R, c+R]$
$\sum a_k(x-c)^k$ *converges* $\forall x \in [c-R, c+R]$

NOTE: For the bottom four cases above, the power series **converges absolutely** on the **open interval** $(c-R, c+R)$.

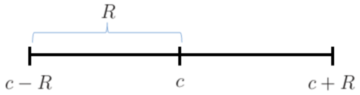
Power Series (Set of Convergence)

Definition

The **set of convergence** for a power series $\sum_{k=0}^{\infty} a_k(x-c)^k$ is the set of all x-values s.t. the power series converges.

Power series $\sum a_k(x-c)^k$ converges...

- ... everywhere $\iff$ set of convergence is $\mathbb{R}$
- ... only for $x = c$ $\iff$ set of convergence is $\{c\}$
- ... $\forall x \in (c-R, c+R)$ $\iff$ set of convergence is $(c-R, c+R)$
- ... $\forall x \in [c-R, c+R)$ $\iff$ set of convergence is $[c-R, c+R)$
- ... $\forall x \in (c-R, c+R]$ $\iff$ set of convergence is $(c-R, c+R]$
- ... $\forall x \in [c-R, c+R]$ $\iff$ set of convergence is $[c-R, c+R]$



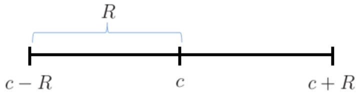
Power Series (Radius of Convergence)

Definition

The **radius of convergence** for a power series $\sum_{k=0}^{\infty} a_k(x-c)^k$ is half the length of the set of convergence.

Power series $\sum a_k(x-c)^k$ converges...

- ... everywhere $\iff$ radius of convergence is ∞
- ... only for $x = c$ $\iff$ radius of convergence is 0
- ... $\forall x \in (c-R, c+R)$ $\iff$ radius of convergence is R
- ... $\forall x \in [c-R, c+R)$ $\iff$ radius of convergence is R
- ... $\forall x \in (c-R, c+R]$ $\iff$ radius of convergence is R
- ... $\forall x \in [c-R, c+R]$ $\iff$ radius of convergence is R



Power Series (Procedure for Convergence)

Given power series $\sum a_k(x-c)^k$,

- To find the **radius of convergence**, use the **Ratio Test** or **Root Test** on $\sum \left|a_k(x-c)^k\right|$:
  - Ratio Test: solve the **inequality** $\lim_{k\to\infty} \left|\frac{a_{k+1}(x-c)^{k+1}}{a_k(x-c)^k}\right| < 1$ for x
  - Root Test: solve the **inequality** $\lim_{k\to\infty} \sqrt[k]{|a_k(x-c)^k|} < 1$ for x
  - If a **true statement** results like $0 < 1$, then **radius of convergence** $R = \infty$
  - If a **false statement** results like $3 < 1$, then **radius of convergence** $R = 0$
- To determine convergence on the **boundary** of interval $(c-R, c+R)$, that is, at $x = c - R$ and $x = c + R$, **test the series for convergence at each endpoint**.
- At this point, the **set of convergence** is known.

Properties of Absolute Value

Proposition

Let $a, b \in \mathbb{R}$. Then:

(i) $|ab| = |a||b|$

(ii) $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$

(iii) $\left|a^k\right| = |a|^k$

(iv) *In general,* $|a+b| \neq |a| + |b|$ *and* $|a-b| \neq |a| - |b|$

PROOF:

(i) $|ab| := \sqrt{(ab)^2} = \sqrt{a^2b^2} = \sqrt{a^2}\sqrt{b^2} := |a||b|$

(ii) $\left|\frac{a}{b}\right| := \sqrt{\left(\frac{a}{b}\right)^2} = \sqrt{\frac{a^2}{b^2}} = \frac{\sqrt{a^2}}{\sqrt{b^2}} := \frac{|a|}{|b|}$

(iii) $\left|a^k\right| := \sqrt{\left(a^k\right)^2} = \left[\left(a^k\right)^2\right]^{1/2} = a^{(k)(2)(1/2)} = \left[\left(a^2\right)^{1/2}\right]^k = \left(\sqrt{a^2}\right)^k := |a|^k$

(iv) Let $a = 1$ and $b = -1$.
Then, $|a+b| = 0 \neq 2 = |a| + |b|$ and $|a-b| = 2 \neq 0 = |a| - |b|$ QED

Power Series (Properties)

A **power series** $\sum_{k=0}^{\infty} a_k (x-c)^k$ with **radius of convergence** $R > 0$:

- Is **infinitely differentiable** on its **interval of absolute convergence**
- Can be **differentiated term by term** on $(c-R, c+R)$:

$$\frac{d}{dx}\left[\sum_{k=0}^{\infty} a_k (x-c)^k\right] = \sum_{k=0}^{\infty} \frac{d}{dx}\left[a_k (x-c)^k\right] = \sum_{k=1}^{\infty} k a_k (x-c)^{k-1}$$

- Can be **integrated term by term** on $(c-R, c+R)$:

$$\int \left(\sum_{k=0}^{\infty} a_k (x-c)^k\right) dx = \sum_{k=0}^{\infty} \left(\int a_k (x-c)^k \, dx\right) = \sum_{k=0}^{\infty} \frac{a_k}{k+1} (x-c)^{k+1} + C$$

$$\int_a^b \left(\sum_{k=0}^{\infty} a_k (x-c)^k\right) dx = \sum_{k=0}^{\infty} \int_a^b a_k (x-c)^k \, dx = \sum_{k=0}^{\infty} \left[\frac{a_k}{k+1} (x-c)^{k+1}\right]_{x=a}^{x=b}$$

- Can be **rearranged without changing its sum** on $(c-R, c+R)$.
- **Behaves like a polynomial** on its interval of absolute convergence.

This means power series can be used to integrate **nonelementary integrals**.

Special Functions

Some **special functions** are defined by power series:

- Bessel Functions: $J_\alpha(x) := \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+\alpha+1)} \left(\frac{x}{2}\right)^{2k+\alpha} \qquad (\alpha \geq 0)$
  - $J_0(x) := \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(k!)^2 2^{2k}} \qquad J_1(x) := \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{k!(k+1)!2^{2k+1}}$
- Error Function: $\text{erf}(x) := \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{k!(2k+1)}$
- Hypergeometric Fcn: $F(a,b;c;x) := \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)} \frac{x^k}{k!}$

Special functions are ”special” in the sense that they tend to show up often in certain branches of mathematics, statistics, physics, and engineering.

IMPORTANT: DO NOT MEMORIZE THESE SPECIAL FUNCTIONS!

Fin.