Taylor Series & Binomial Series

Calculus II

Josh Engwer

TTU

09 April 2014

Continuity & Differentiability of a Function (Notation)

Definition

Given function $f(x)$ and set $S \subseteq \mathbb{R}$.   Then:

$$f \in C(S) \iff f \text{ is continuous on set } S$$

$$f \in C^1(S) \iff f, f' \in C(S) \implies f \text{ is differentiable on set } S$$

$$f \in C^2(S) \iff f, f', f'' \in C(S) \implies f \text{ is twice-differentiable on set } S$$

$$f \in C^3(S) \iff f, f', f'', f''' \in C(S) \implies f \text{ is 3-times differentiable on set } S$$

$$f \in C^4(S) \iff f, f', f'', f''', f^{(4)} \in C(S) \implies f \text{ is 4-times differentiable on set } S$$

$$f \in C^\infty(S) \iff \underline{\text{all derivatives}} \text{ exist and are continuous on set } S \implies f \text{ is } \underline{\text{infinitely differentiable}} \text{ on set } S$$

Taylor Series (Motivation)

TASK: Find a power series representation for a function $f(x) \in C^{\infty}$:

$$f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

What are the values of the coefficients $a_0, a_1, a_2, a_3, \ldots$???

Taylor Series (Motivation)

$$f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

Observe that $f(c) = a_0 \implies a_0 = f(c)$

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Observe that $f(c) = a_0 \implies a_0 = f(c)$

$$f'(x) = \sum_{k=0}^{\infty} k a_k (x-c)^{k-1} = a_1 + 2a_2(x-c) + 3a_3(x-c)^2 + \cdots$$

Observe that $f'(c) = a_1 \implies a_1 = \frac{f'(c)}{1!}$

Taylor Series (Motivation)

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Observe that $f'(c) = a_1 \implies a_1 = \dfrac{f'(c)}{1!}$

$$f''(x) = \sum_{k=0}^{\infty} k(k-1) a_k (x-c)^{k-2} = 2a_2 + 6a_3(x-c) + \cdots$$

Observe that $f''(c) = 2a_2 \implies a_2 = \dfrac{f''(c)}{2} = \dfrac{f''(c)}{2!}$

Taylor Series (Motivation)

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Observe that $f''(c) = 2a_2 \implies a_2 = \dfrac{f''(c)}{2} = \dfrac{f''(c)}{2!}$

$$f'''(x) = \sum_{k=0}^{\infty} k(k-1)(k-2)a_k(x-c)^{k-3} = 6a_3 + \cdots$$

Observe that $f'''(c) = 6a_3 \implies a_3 = \dfrac{f'''(c)}{6} = \dfrac{f'''(c)}{3!}$

Taylor Series (Motivation)

$$f(x) = \sum_{k=0}^{\infty} a_k (x-c)^k = a_0 + a_1(x-c) + a_2(x-c)^2 + a_3(x-c)^3 + \cdots$$

$$f(c) = a_0 \implies a_0 = f(c)$$

$$f'(c) = a_1 \implies a_1 = \frac{f'(c)}{1!}$$

$$f''(c) = 2a_2 \implies a_2 = \frac{f''(c)}{2} = \frac{f''(c)}{2!}$$

$$f'''(c) = 6a_3 \implies a_3 = \frac{f'''(c)}{6} = \frac{f'''(c)}{3!}$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$f^{(k)}(c) = k(k-1)(k-2)\cdots(3)(2)a_k \implies a_k = \frac{f^{(k)}(c)}{k!}$$

Taylor Series (Definition)

Definition

Let $f \in C^\infty(c-R, c+R)$. Then the **Taylor series about** $x = c$ has the form

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k =$$

$$f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \frac{f^{(4)}(c)}{4!}(x-c)^4 + \cdots$$

which is a **power series** that **converges absolutely** to $f(x)\ \ \forall x \in (c-R, c+R)$ where R is the **radius of convergence**.

A **Maclaurin series** is just a **Taylor series about** $x = 0$:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$

Existence & Uniqueness of a Taylor Series Representation

Theorem

(Taylor Series Representation Theorem)

Let $f \in C^{\infty}(c-R, c+R)$.

Then $f(x)$ *has a unique power series representation:*

$$f(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3 + \frac{f^{(4)}(c)}{4!}(x-c)^4 + \cdots$$

i.e. the unique power series representation for $f(x)$ *is the* ***Taylor series****.*

PROOF: See the textbook if interested.

Construction of a Taylor Series (Toolkit)

- Computing $f(c), f'(c), f''(c), f'''(c), f^{(4)}(c), \ldots$ (Always available)
- Clever **substitution** into a known Taylor series
- Clever manipulation of a **geometric series**
- **Differentiating** a known Taylor series
- **Integrating** a known Taylor series
- Clever use of **trig identities**
- **Multiplying** a Taylor series by a **monomial**
- **Dividing** a Taylor series by a **monomial**
- **Multiplying** two Taylor series
- **Dividing** two Taylor series (this is subtle, so not considered here)
- Clever manipulation of a **Binomial series** (see next slide)

Binomial Series

Corollary

*The **Binomial Series** is*

$$(1+x)^{\alpha} = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$$

where $\alpha \in \mathbb{R}$ *and* $|x| < 1$

Binomial Series

Corollary

The ***Binomial Series*** *is*

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$$

where $\alpha \in \mathbb{R}$ *and* $|x| < 1$

PROOF: Construct the **Taylor Series** about $x = 0$ of $f(x) = (1+x)^\alpha$:

$$\begin{array}{rcl@{\quad}c@{\quad}rcl}
f(x) & = & (1+x)^\alpha & \Longrightarrow & f(0) & = & 1 \\
f'(x) & = & \alpha(1+x)^{\alpha-1} & \Longrightarrow & f'(0) & = & \alpha \\
f''(x) & = & \alpha(\alpha-1)(1+x)^{\alpha-2} & \Longrightarrow & f''(0) & = & \alpha(\alpha-1) \\
f'''(x) & = & \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3} & \Longrightarrow & f'''(0) & = & \alpha(\alpha-1)(\alpha-2) \\
\vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

$$\Longrightarrow (1+x)^\alpha = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

$$\Longrightarrow (1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$$

Binomial Series

Corollary

*The **Binomial Series** is*

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots$$

where $\alpha \in \mathbb{R}$ *and* $|x| < 1$

PROOF: Observe that the Binomial Series is a **power series** $\sum_{k=0}^{\infty} a_k x^k$

with $a_k = \dfrac{\alpha(\alpha-1)(\alpha-2)\cdots[\alpha-(k-1)]}{k!}$.

Then it converges provided $\lim_{k\to\infty}\left|\dfrac{a_{k+1}x^{k+1}}{a_k x^k}\right| < 1$:

$$\implies \lim_{k\to\infty}\left|\frac{\alpha(\alpha-1)\cdots[\alpha-(k-1)]\,(\alpha-k)}{(k+1)!}\cdot\frac{k!}{\alpha(\alpha-1)\cdots[\alpha-(k-1)]}\right||x| < 1$$

$$\implies \lim_{k\to\infty}\left|\frac{\alpha-k}{k+1}\right||x| < 1 \implies \lim_{k\to\infty}\frac{\left|\frac{\alpha}{k}-1\right|}{1+\frac{1}{k}}|x| < 1 \implies \frac{|0-1|}{1+0}|x| < 1$$

$$\implies |x| < 1$$

Binomial Series

Corollary

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where $\alpha \in \mathbb{R}$ *and* $|x| < 1$

PROOF: Finally, check convergence at the endpoints $x = 1$ and $x = -1$:

Take **Advanced Calculus.**

QED

Applications of Taylor Series

- Finding **limits** that would require too many iterations of **L'Hôpital's Rule**
- Evaluation of **nonelementary integrals**
- Solution to advanced **differential equations** (not considered here)
- **Polynomial approximation** of a complicated function (not considered)
- Central to functions of **complex variables** (not considered here)

A Note about Convergence of a Taylor Series

- It's possible that the **set of convergence** of a function is a **single point**:
  - e.g. Let $f(x) = \begin{cases} e^{-1/x^2} & \text{, if } x \neq 0 \\ 0 & \text{, if } x = 0 \end{cases}$
  - Then the Taylor series about $x = 0$ for $f(x)$ is the **zero function**:

$$0 + \frac{0}{1!}x + \frac{0}{2!}x^2 + \frac{0}{3!}x^3 + \frac{0}{4!}x^4 + \cdots$$

  - Therefore, the **set of convergence** for $f(x)$ is $\{0\}$.
- It's possible that $f(x)$ exists for x-values outside of the **set of convergence** of its Taylor series.
  - e.g. The **Taylor series about** $x = 0$ for $\ln(1+x)$ **converges** for all $x \in (-1, 1]$, yet $\ln(1+2)$ is defined!!
  - i.e. A Taylor series for $f(x)$ is a **very poor approximation** to $f(x)$ for all x **outside the set of convergence**.

Taylor's Remainder Theorem

Theorem

(Taylor's Remainder Theorem)

Let $f \in C^{n+1}(a,b)$ *where* $c \in (a,b)$*. Then*

$$f(x) = T_n(x) + R_n(x) \qquad \forall x \in (a,b)$$

where

$$T_n(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

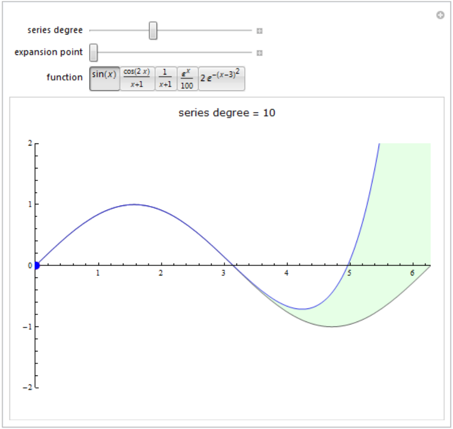
is the n^{th} ***Taylor polynomial*** *and*

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}$$ *is the* n^{th} ***Taylor Remainder***

where ξ *depends on* x *and lies between* c *and* x*.*

REMARK: ξ is the lowercase Greek letter "xi" (pronounced $kuh - SEE$)
PROOF: Take **Advanced Calculus**.
This is used in **Numerical Analysis**, and so will not be considered here.

(DEMO) TAYLOR APPROXIMATIONS OF FUNCTIONS (Click below):



Fin

Fin.