Vectors: Introduction

Calculus II

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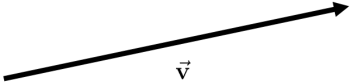
TTU

23 April 2014

Vectors & Scalars (Definition)



Definition

A **vector** $\vec{v}$ is a quantity that bears **both magnitude and direction**.

Examples of vectors: displacement, velocity, force, angular momentum, ...

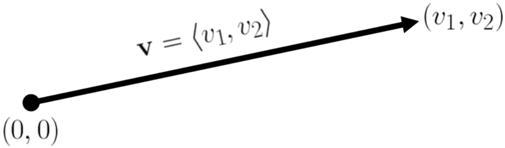
Definition

A **scalar** t is a quantity that bears **only magnitude**.

Examples of scalars: time, temperature, distance, speed, area, volume, ...

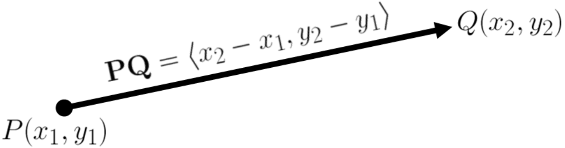
2D Vector (Definition)

Vector $\mathbf{v}$ with **horizontal component** v_1 and **vertical component** v_2:



2D Vector (Definition)

Vector $\mathbf{PQ}$ with **initial point** P and **terminal point** Q:



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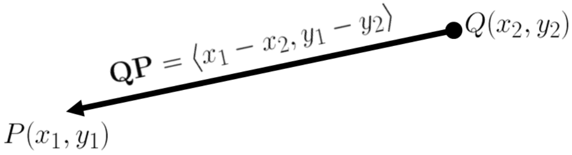


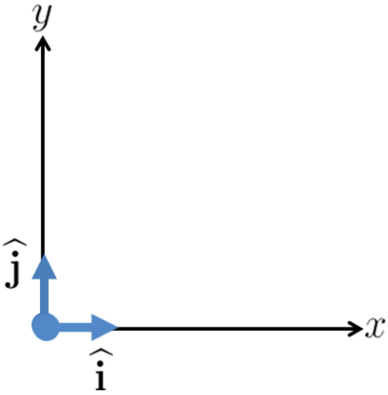
2D Vector (Representation)

2D Space: $\mathbb{R}^2$ (say "R-Two")

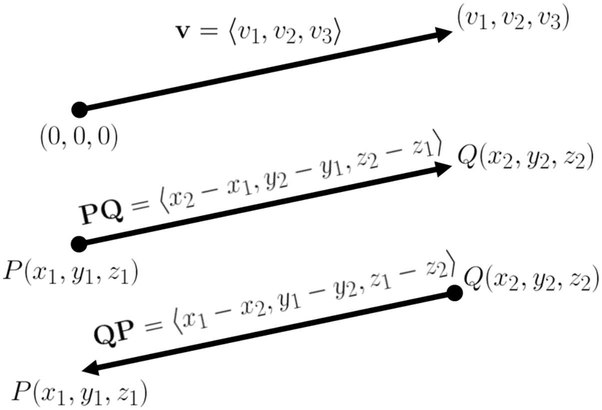
Component Form: $\mathbf{v} = \langle v_1, v_2 \rangle$

Basis Form: $\mathbf{v} = v_1\widehat{\mathbf{i}} + v_2\widehat{\mathbf{j}}$

$\left[\text{Basis Vectors } \widehat{\mathbf{i}} := \langle 1, 0 \rangle, \widehat{\mathbf{j}} := \langle 0, 1 \rangle \right]$



3D Vector (Definition)



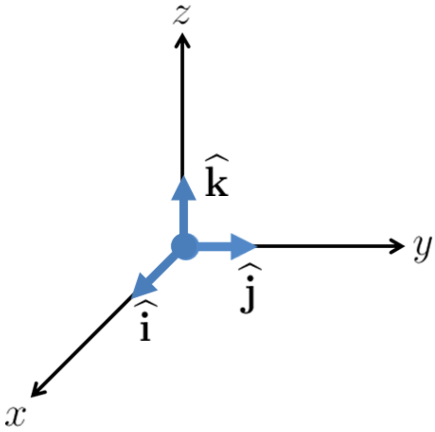
3D Vector (Representation)

3D Space: $\mathbb{R}^3$ (say "R-Three")

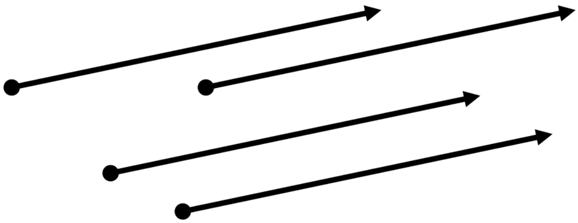
Component Form: $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$

Basis Form: $\mathbf{v} = v_1\widehat{\mathbf{i}} + v_2\widehat{\mathbf{j}} + v_3\widehat{\mathbf{k}}$

[Basis Vectors $\widehat{\mathbf{i}} := \langle 1, 0, 0 \rangle, \widehat{\mathbf{j}} := \langle 0, 1, 0 \rangle, \widehat{\mathbf{k}} := \langle 0, 0, 1 \rangle$]

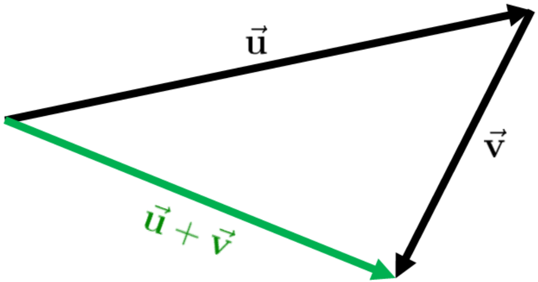


Vectors (Translation Invariance)



- Two vectors are **equal** $\iff$ their components are equal $\iff$ they both bear the same magnitude & same direction.
- Merely **translating a vector** (as illustrated above) **does not change it.**

Vector Algebra (Addition)



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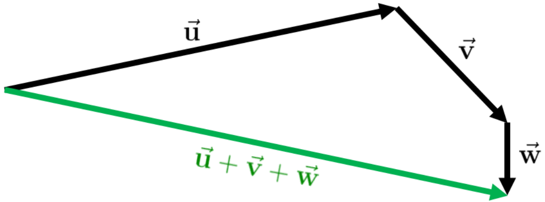

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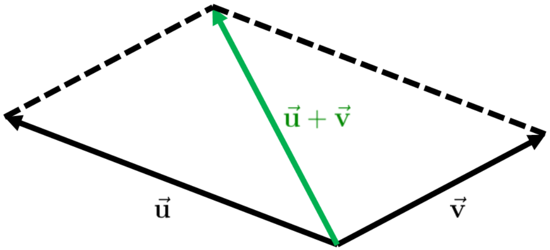
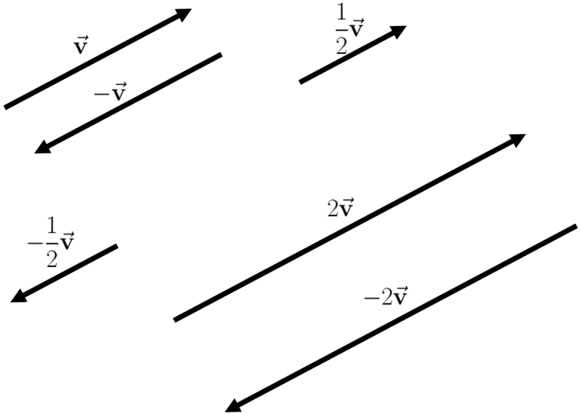
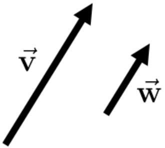
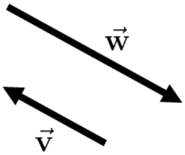


Vector Algebra (Scalar Multiplication)



Parallel Vectors (Definition)



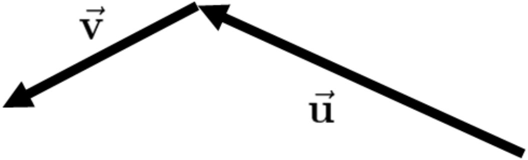


Definition

Nonzero Vectors $\vec{\mathbf{v}}, \vec{\mathbf{w}}$ are **parallel** $\iff$ $\vec{\mathbf{v}} \,||\, \vec{\mathbf{w}}$ $\iff$ $\vec{\mathbf{v}} = k\vec{\mathbf{w}}$ for some $k \in \mathbb{R}$.

Vector Algebra (Subtraction)

Find $\mathbf{u} - \mathbf{v}$.



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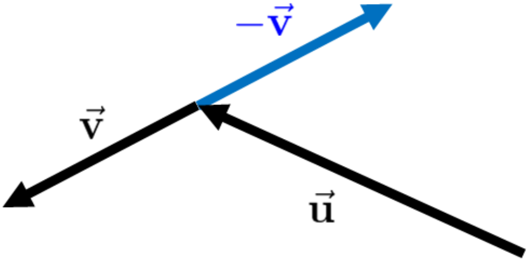

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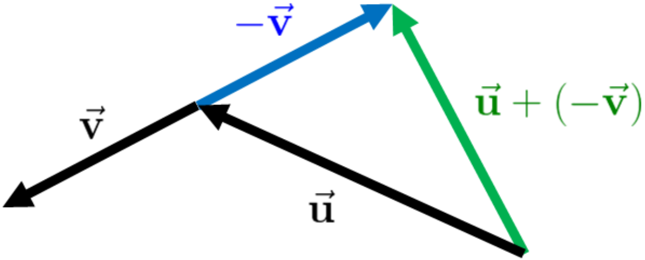

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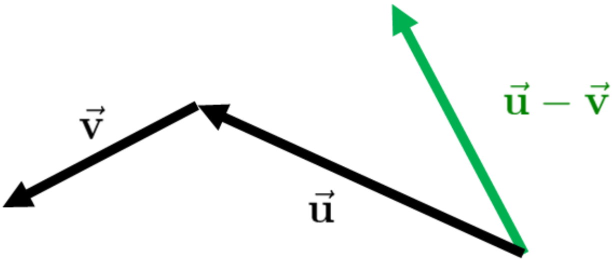

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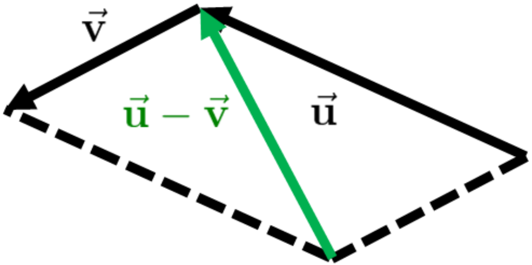

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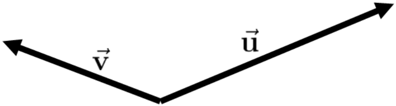

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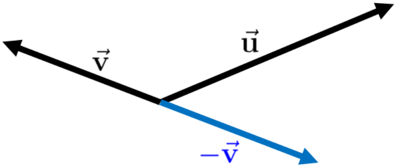

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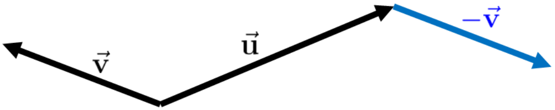

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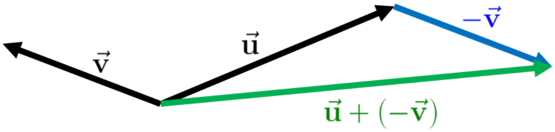

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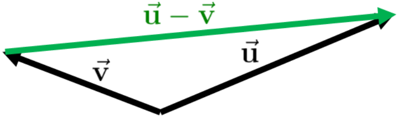


Vector Algebra (Operations)

Let vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$ and scalar $k \in \mathbb{R}$. Then:

- $\mathbf{u} + \mathbf{v} = \langle u_1, u_2 \rangle + \langle v_1, v_2 \rangle = \langle u_1 + v_1, u_2 + v_2 \rangle$
- $\mathbf{u} - \mathbf{v} = \langle u_1, u_2 \rangle - \langle v_1, v_2 \rangle = \langle u_1 - v_1, u_2 - v_2 \rangle$
- $k\mathbf{v} = k\langle v_1, v_2 \rangle = \langle kv_1, kv_2 \rangle$

Let vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ and scalar $k \in \mathbb{R}$. Then:

- $\mathbf{u} + \mathbf{v} = \langle u_1, u_2, u_3 \rangle + \langle v_1, v_2, v_3 \rangle = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$
- $\mathbf{u} - \mathbf{v} = \langle u_1, u_2, u_3 \rangle - \langle v_1, v_2, v_3 \rangle = \langle u_1 - v_1, u_2 - v_2, u_3 - v_3 \rangle$
- $k\mathbf{v} = k\langle v_1, v_2, v_3 \rangle = \langle kv_1, kv_2, kv_3 \rangle$

Zero Vector:

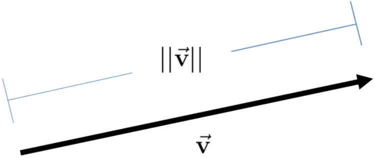
- $\vec{\mathbf{0}} := \langle 0, 0 \rangle$
- $\vec{\mathbf{0}} := \langle 0, 0, 0 \rangle$

Vector Algebra (Properties)

Let vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \{\mathbb{R}^2, \mathbb{R}^3\}$ and scalars $s, t \in \mathbb{R}$. Then:

- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
- $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- $(st)\mathbf{u} = s(t\mathbf{u}) = t(s\mathbf{u})$
- $\mathbf{u} + \vec{\mathbf{0}} = \mathbf{u}$
- $\mathbf{u} - \mathbf{u} = \mathbf{u} + (-\mathbf{u}) = \vec{\mathbf{0}}$
- $(s + t)\mathbf{u} = s\mathbf{u} + t\mathbf{u}$
- $s(\mathbf{u} + \mathbf{v}) = s\mathbf{u} + s\mathbf{v}$

Vectors (Norm)



Definition

The **norm** of a 2D vector $\mathbf{v} = \langle v_1, v_2 \rangle$ is defined to be

$$||\mathbf{v}|| := \sqrt{v_1^2 + v_2^2}$$

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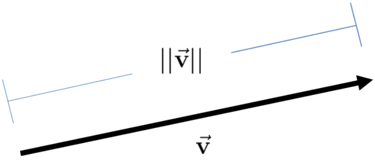

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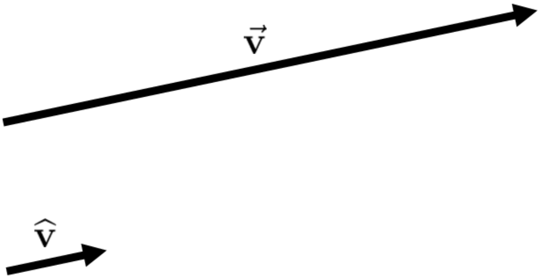
PROOF: Use Pythagorean's Theorem.

Unit Vectors & Direction Vectors



Definition

A **unit vector** $\widehat{\mathbf{v}}$ is a vector with **norm one**.
A **direction vector** for vector $\mathbf{v}$ is defined to be

$$\widehat{\mathbf{v}} := \frac{\mathbf{v}}{||\mathbf{v}||}$$

Fin.