Vectors: Dot Products & Projections

Calculus II

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TTU

28 April 2014

Dot Product (Definition)

Definition

Dot Product in $\mathbb{R}^2$:
The **dot product** of vectors $\vec{\mathbf{v}} = \langle v_1, v_2 \rangle$ and $\vec{\mathbf{w}} = \langle w_1, w_2 \rangle$ is defined by:

$$\mathbf{v} \cdot \mathbf{w} := v_1 w_1 + v_2 w_2$$

Definition

Dot Product in $\mathbb{R}^3$:
The **dot product** of vectors $\vec{\mathbf{v}} = \langle v_1, v_2, v_3 \rangle$ and $\vec{\mathbf{w}} = \langle w_1, w_2, w_3 \rangle$ is defined by:

$$\mathbf{v} \cdot \mathbf{w} := v_1 w_1 + v_2 w_2 + v_3 w_3$$

REMARKS:

- Notice that the dot product $\mathbf{v} \cdot \mathbf{w}$ is a **scalar**.
- Going forward, the focus will be on **3-D vectors** $\Big($e.g. $\mathbf{v} = \langle v_1, v_2, v_3 \rangle\Big)$

Dot Product (Properties)

Let vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ and scalar $c \in \mathbb{R}$. Then:

- $\mathbf{v} \cdot \mathbf{v} = ||\mathbf{v}||^2$
- $\vec{\mathbf{0}} \cdot \mathbf{v} = \mathbf{v} \cdot \vec{\mathbf{0}} = 0$
- $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$
- $c(\mathbf{v} \cdot \mathbf{w}) = (c\mathbf{v}) \cdot \mathbf{w} = \mathbf{v} \cdot (c\mathbf{w})$
- $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

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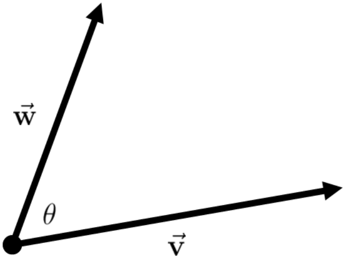
- $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$

PROOF: Let $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$. Then:

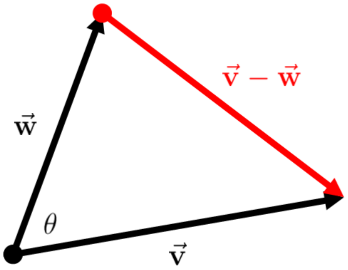
$$\mathbf{v} \cdot \mathbf{v} = \langle v_1, v_2, v_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle = v_1^2 + v_2^2 + v_3^2 = \left(\sqrt{v_1^2 + v_2^2 + v_3^2}\right)^2 = ||\mathbf{v}||^2 \quad \text{QED}$$

Dot Product (Geometric Formula Derivation)



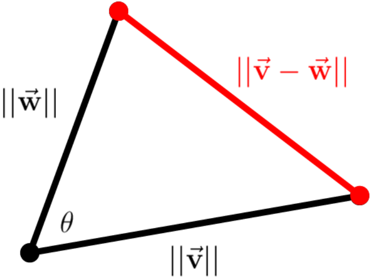
- Given vectors $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$.

Dot Product (Geometric Formula Derivation)



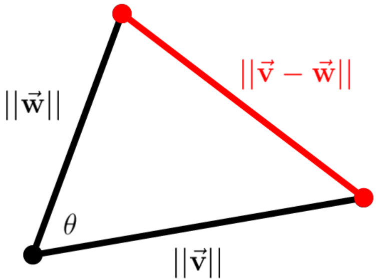
- Form vector $\mathbf{v} - \mathbf{w} = \langle v_1 - w_1, v_2 - w_2, v_3 - w_3 \rangle$.

Dot Product (Geometric Formula Derivation)



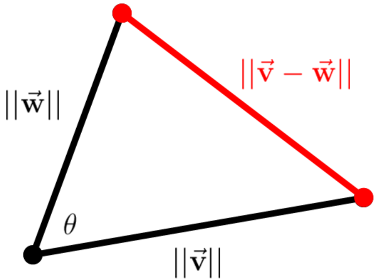
- Take norms of all three vectors and consider the resulting **triangle**.

Dot Product (Geometric Formula Derivation)



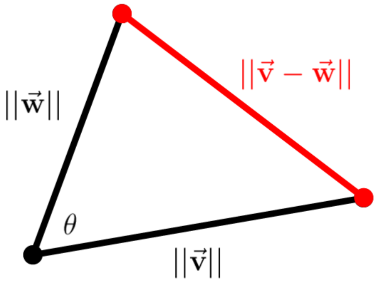
Trig Review: What result relates all three norms and the angle θ??

Dot Product (Geometric Formula Derivation)



Law of Cosines: $||\mathbf{v} - \mathbf{w}||^2 = ||\mathbf{v}||^2 + ||\mathbf{w}||^2 - 2||\mathbf{v}||||\mathbf{w}|| \cos\theta$

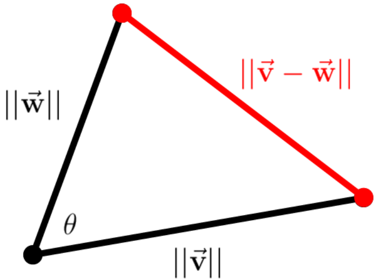
Dot Product (Geometric Formula Derivation)



Solve for the cosine term:

$$\cos\theta = \frac{||\mathbf{v}||^2 + ||\mathbf{w}||^2 - ||\mathbf{v} - \mathbf{w}||^2}{2||\mathbf{v}||||\mathbf{w}||}$$

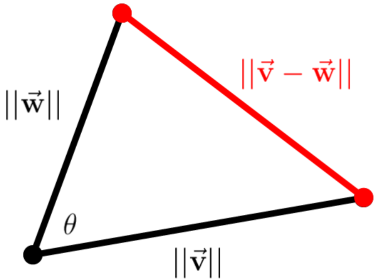
Dot Product (Geometric Formula Derivation)



Write all norms in the numerator in terms of vector components:

$$\cos\theta = \frac{v_1^2 + v_2^2 + v_3^2 + w_1^2 + w_2^2 + w_3^2 - \left[(v_1 - w_1)^2 + (v_2 - w_2)^2 + (v_3 - w_3)^2\right]}{2||\mathbf{v}||||\mathbf{w}||}$$

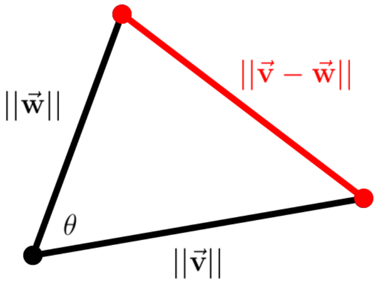
Dot Product (Geometric Formula Derivation)



Simplify numerator:

$$\cos\theta = \frac{2v_1w_1 + 2v_2w_2 + 2v_3w_3}{2||\mathbf{v}||||\mathbf{w}||}$$

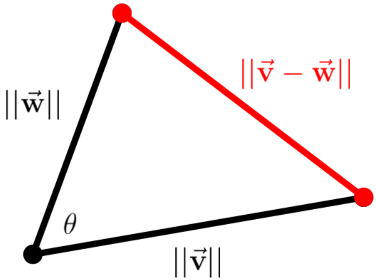
Dot Product (Geometric Formula Derivation)



Simplify fraction:

$$\cos\theta = \frac{v_1 w_1 + v_2 w_2 + v_3 w_3}{||\mathbf{v}|| ||\mathbf{w}||}$$

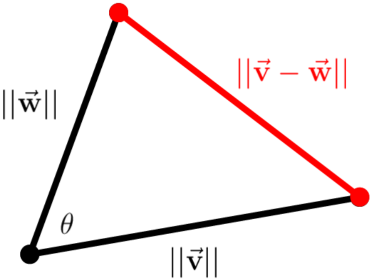
Dot Product (Geometric Formula Derivation)



Realize that the numerator is a dot product:

$$\cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{v}||||\mathbf{w}||}$$

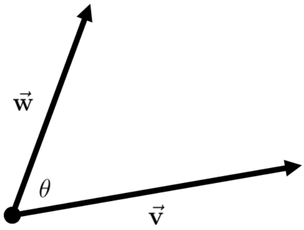
Dot Product (Geometric Formula Derivation)



Solve for the dot product:

$$\mathbf{v} \cdot \mathbf{w} = ||\mathbf{v}|| ||\mathbf{w}|| \cos\theta$$

Dot Product (Coordinate-Free Definition)



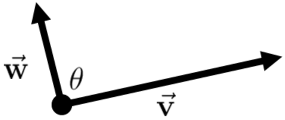
Definition

Let θ be the smallest positive angle between vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. Then:

$$\mathbf{v} \cdot \mathbf{w} = ||\mathbf{v}||\,||\mathbf{w}|| \cos\theta \qquad \text{where } \theta \in [0, \pi]$$

- Alternative notation for the angle between vectors $\mathbf{v}, \mathbf{w}$: θ_{vw}

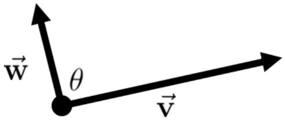
Dot Product (Orthogonality)



Theorem

Vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ *are* ***orthogonal*** $\iff \mathbf{v} \perp \mathbf{w} \iff \mathbf{v} \cdot \mathbf{w} = 0$

Dot Product (Orthogonality)



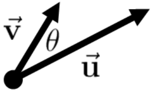
Theorem

Vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ *are* ***orthogonal*** $\iff \mathbf{v} \perp \mathbf{w} \iff \mathbf{v} \cdot \mathbf{w} = 0$

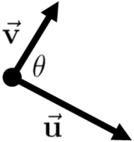
PROOF:

$\mathbf{v}, \mathbf{w}$ are **orthogonal** $\iff \theta = \pi/2 \iff \mathbf{v} \cdot \mathbf{w} = ||\mathbf{v}||||\mathbf{w}|| \cos(\pi/2) = 0$ QED

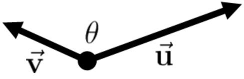
Dot Product (Geometric Interpretation)



θ is acute
$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} > 0$

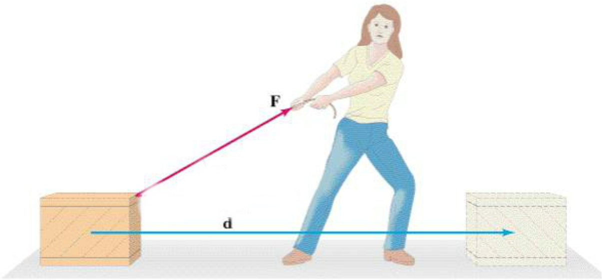


θ is 90°
$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 0$



θ is obtuse
$\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} < 0$

Dot Product (Work)



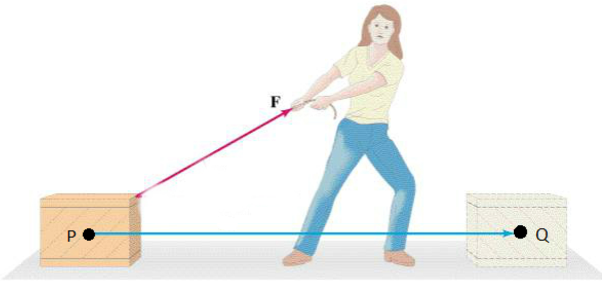
Definition

The **work**, W, done by a constant force $\vec{\mathbf{F}}$ on an object displacing it by $\vec{\mathbf{d}}$ is:

$$W := \vec{\mathbf{F}} \cdot \vec{\mathbf{d}}$$

PROOF: Take Physics (Mechanics)

Dot Product (Work)



Definition

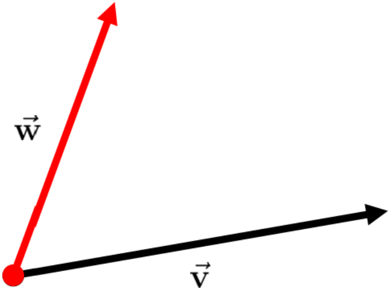
The **work**, W, done by a constant force $\vec{\mathbf{F}}$ on an object moving it from P to Q is:

$$W := \vec{\mathbf{F}} \cdot \mathbf{PQ}$$

PROOF: Take Physics (Mechanics)

Projection (Example 1)

Project $\mathbf{w}$ onto $\mathbf{v}$.



Projection (Example 1)

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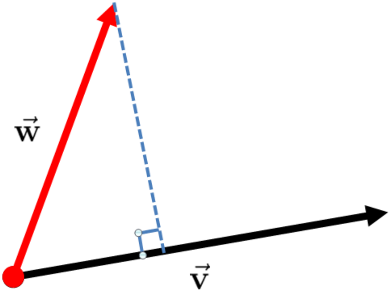
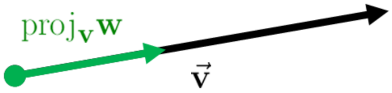


Drop perpendicular line from $\mathbf{w}$ to $\mathbf{v}$.

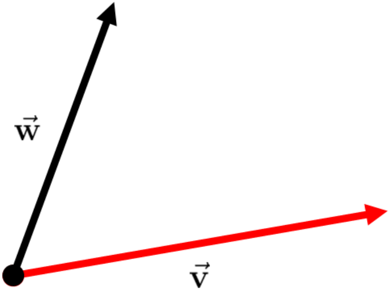
Projection (Example 1)

Project $\mathbf{w}$ onto $\mathbf{v}$.



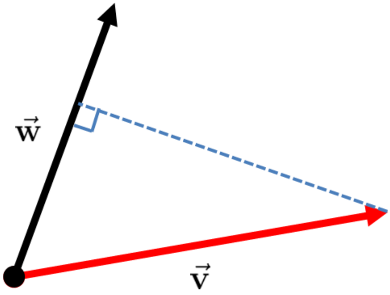
Projection (Example 2)

Project **v** onto **w**.



Projection (Example 2)

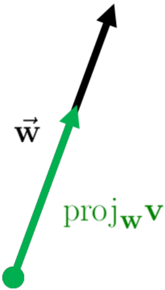
Project $\mathbf{v}$ onto $\mathbf{w}$.



Drop perpendicular line from $\mathbf{v}$ to $\mathbf{w}$.

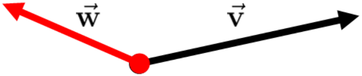
Projection (Example 2)

Project $\mathbf{v}$ onto $\mathbf{w}$.



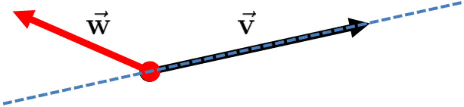
Projection (Example 3)

Project $\mathbf{w}$ onto $\mathbf{v}$.



Projection (Example 3)

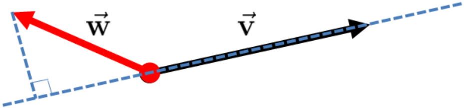
Project $\mathbf{w}$ onto $\mathbf{v}$.



Draw line extension through $\mathbf{v}$.

Projection (Example 3)

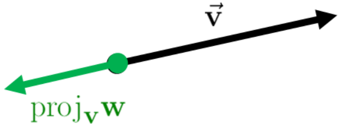
Project $\mathbf{w}$ onto $\mathbf{v}$.



Drop perpendicular line from $\mathbf{w}$ to line extension.

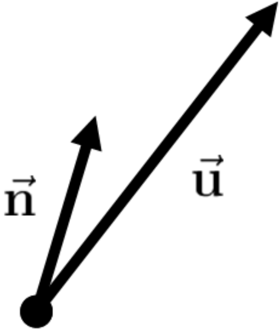
Projection (Example 3)

Project $\mathbf{w}$ onto $\mathbf{v}$.



Projection (Example 4)

Project $\mathbf{u}$ onto $\mathbf{n}$.



Projection (Example 4)

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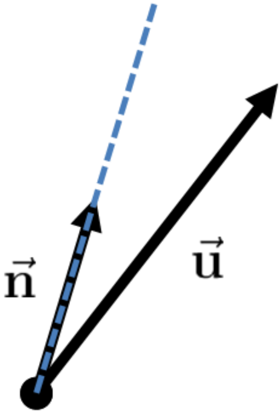
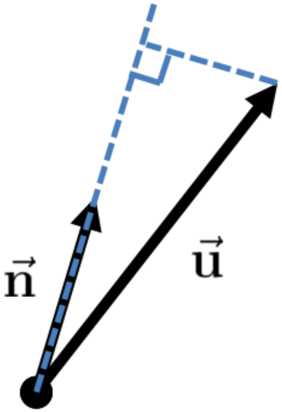


Draw line extension through $\mathbf{n}$.

Projection (Example 4)

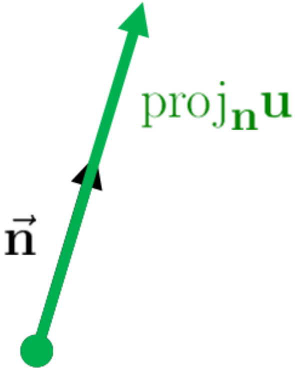
Project **u** onto **n**.



Drop perpendicular line from **u** to line extension.

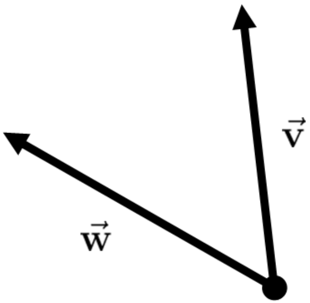
Projection (Example 4)

Project $\mathbf{u}$ onto $\mathbf{n}$.



Projection (Formula Derivation)

Determine a formula for $\text{proj}_{\mathbf{w}}\mathbf{v}$, the **projection** of vector $\mathbf{v}$ onto vector $\mathbf{w}$.



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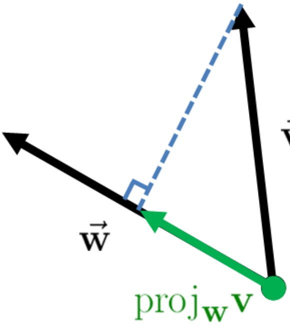
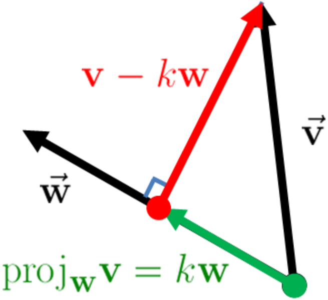


Notice that $(\text{proj}_{\mathbf{w}}\mathbf{v}) \;||\; \mathbf{w} \implies \text{proj}_{\mathbf{w}}\mathbf{v} = k\mathbf{w}$, where $k \in \mathbb{R}$.

Projection (Formula Derivation)

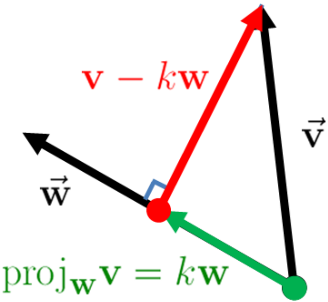
Determine value of scalar k in terms of given vectors $\mathbf{v}$ & $\mathbf{w}$.



Form vector $\mathbf{v} - k\mathbf{w}$.

Projection (Formula Derivation)

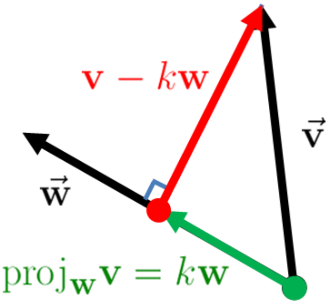
Determine value of scalar k in terms of given vectors $\mathbf{v}$ & $\mathbf{w}$.



Notice that $(\mathbf{v} - k\mathbf{w}) \perp \mathbf{w}$.

Projection (Formula Derivation)

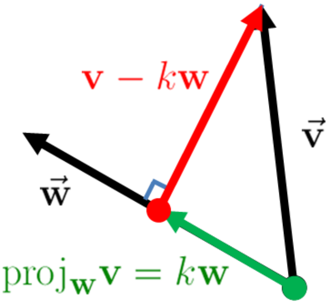
Determine value of scalar k in terms of given vectors $\mathbf{v}$ & $\mathbf{w}$.



$$\implies (\mathbf{v} - k\mathbf{w}) \cdot \mathbf{w} = 0$$

Projection (Formula Derivation)

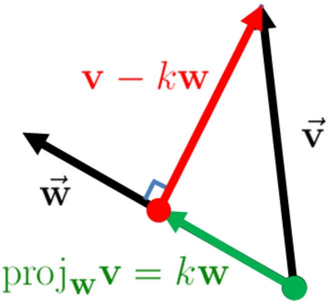
Determine value of scalar k in terms of given vectors $\mathbf{v}$ & $\mathbf{w}$.



$$\implies \mathbf{v} \cdot \mathbf{w} - (k\mathbf{w}) \cdot \mathbf{w} = 0$$

Projection (Formula Derivation)

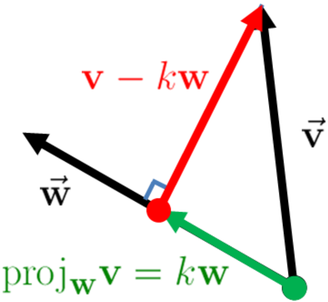
Determine value of scalar k in terms of given vectors $\mathbf{v}$ & $\mathbf{w}$.



$$\implies \mathbf{v} \cdot \mathbf{w} - k(\mathbf{w} \cdot \mathbf{w}) = 0$$

Projection (Formula Derivation)

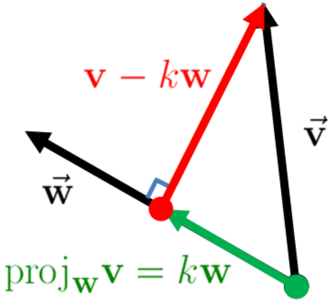
Determine value of scalar k in terms of given vectors $\mathbf{v}$ & $\mathbf{w}$.



$$\implies \mathbf{v} \cdot \mathbf{w} = k(\mathbf{w} \cdot \mathbf{w})$$

Projection (Formula Derivation)

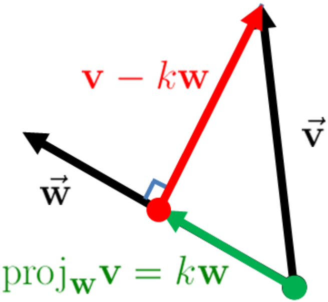
Determine value of scalar k in terms of given vectors $\mathbf{v}$ & $\mathbf{w}$.



$$\implies k = \frac{\mathbf{v} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}}$$

Projection (Formula Derivation)

Determine value of scalar k in terms of given vectors $\mathbf{v}$ & $\mathbf{w}$.



$$\implies k = \frac{\mathbf{v} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}}$$

$$\implies \text{proj}_{\mathbf{w}}\mathbf{v} = k\mathbf{w} = \left(\frac{\mathbf{v} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}} \right) \mathbf{w}$$

Projection (Formula)

Definition

The **(vector) projection** of vector $\mathbf{v}$ onto vector $\mathbf{w}$ is defined by:

$$\text{proj}_{\mathbf{w}}\mathbf{v} := \left(\frac{\mathbf{v} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}}\right) \mathbf{w}$$

Definition

The **scalar projection** of vector $\mathbf{v}$ onto vector $\mathbf{w}$ is defined by:

$$\text{comp}_{\mathbf{w}}\mathbf{v} := \pm||\text{proj}_{\mathbf{w}}\mathbf{v}||$$

The sign is **positive** if $\mathbf{v} \cdot \mathbf{w} \geq 0$ and **negative** if $\mathbf{v} \cdot \mathbf{w} < 0$

REMARKS:

- "Projection" means "**vector** projection."
- I'll never say "scalar projection." Instead, I'll say "norm of the projection."
- I'll never write $\text{comp}_{\mathbf{w}}\mathbf{v}$. Instead, I'll write $\pm||\text{proj}_{\mathbf{w}}\mathbf{v}||$.

Fin

Fin.