Essential Logic & Set Theory

LOGIC NOTATION: Let $\mathcal{P}$,$\mathcal{Q}$ denote **mathematical statements**

- (**Logical Connectives**) AND, OR, NOT (they function as you would expect them to)
- (**Implication**) $\mathcal{P} \implies \mathcal{Q}$ reads "If $\mathcal{P}$ then $\mathcal{Q}$" or "$\mathcal{P}$ implies $\mathcal{Q}$"
- (**Logical Equivalence**) $\mathcal{P} \iff \mathcal{Q}$ reads "$\mathcal{P}$ is equivalent to $\mathcal{Q}$" or "$\mathcal{P}$ if and only if $\mathcal{Q}$" or "$\mathcal{P} \implies \mathcal{Q}$ AND $\mathcal{Q} \implies \mathcal{P}$"
- **Converse** of ($\mathcal{P} \implies \mathcal{Q}$) is ($\mathcal{Q} \implies \mathcal{P}$)
- **Contrapositive** of ($\mathcal{P} \implies \mathcal{Q}$) is (NOT $\mathcal{Q} \implies$ NOT $\mathcal{P}$)
- (**Contradiction**) $\mathcal{P}$ AND (NOT $\mathcal{P}$) (In other words, a math statement that's both true and false, which is absurd!)
- e.g. Let $\mathcal{P}$ be "$x + 2 = 5$" and $\mathcal{Q}$ be "$x = 3$". Then, implication ($\mathcal{P} \implies \mathcal{Q}$) reads "If $x + 2 = 5$, then $x = 3$"
  its converse is "If $x = 3$, then $x + 2 = 5$" and its contrapositive is "If $x \neq 3$, then $x + 2 \neq 5$"

SET NOTATION: Sets contain elements, but never duplicate elements.

- $x \in A \iff$ "x is an element of the set A" $\qquad x \notin A \iff$ "x is NOT an element of the set A"
- $A \subset B \iff$ "set A is a subset of B" $\iff$ "set A is contained in set B" $\iff (x \in A \implies x \in B)$
- $x \in A \cup B \iff x \in A$ or $x \in B$ (**union** of two sets)
- $x \in A \cap B \iff x \in A$ and $x \in B$ (**intersection** of two sets)
- $x \in A \setminus B \iff x \in A$ and $x \notin B$ (**subtraction** of two sets)
- A, B are **disjoint sets** $\iff A \cap B = \emptyset$ (In other words, disjoint sets have no elements in common.)
- e.g. $A = \{1, 2\}, B = \{2, 3, 4\}, C = \{5, 7\} \implies A \cup B = \{1, 2, 3, 4\}, A \cap B = \{2\}, A \setminus B = \{1\}, B \cap C = \emptyset, B \setminus C = B$

SPECIAL SETS:

- $\emptyset \equiv$ the **empty set**
- $\mathbb{N} \equiv$ the set of **natural numbers** $:= \{1, 2, 3, 4, 5, \cdots\}$
- $\mathbb{Z} \equiv$ the set of **integers** $\qquad \mathbb{Z}_+ \equiv$ the set of **positive integers** $\qquad \mathbb{Z}_- \equiv$ the set of **negative integers**
- $\mathbb{Q} \equiv$ the set of **rationals** $\qquad \mathbb{Q}_+ \equiv$ the set of **positive rationals** $\qquad \mathbb{Q}_- \equiv$ the set of **negative rationals**
- $\mathbb{R} \equiv$ the set of **real numbers** $\qquad \mathbb{R}_+ \equiv$ the set of **positive reals** $\qquad \mathbb{R}_- \equiv$ the set of **negative reals**
- Relationship among these sets of numbers : $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$

INTERVALS:

- $x \in (a, b) \iff a < x < b$ (**Open interval**)
- $x \in [a, b] \iff a \leq x \leq b$ (**Closed interval**)
- $x \in (a, b] \iff a < x \leq b$
- $x \in [a, b) \iff a \leq x < b$
- $x \in (-\infty, \infty) \iff x \in \mathbb{R} \iff$ "x is any real number"
- e.g. "n is any integer between $-\sqrt{3}$ and 3.5" $\iff n \in \mathbb{Z} \cap (-\sqrt{3}, 3.5) \iff n \in \{-1, 0, 1, 2, 3\}$
- e.g. "x is any positive real number" $\iff x \in \mathbb{R}_+ \iff x \in (0, \infty) \iff x > 0$
- e.g. "y is any real number except π and 100" $\iff y \in \mathbb{R} \setminus \{\pi, 100\} \iff y \in (-\infty, \pi) \cup (\pi, 100) \cup (100, \infty)$

LOGIC QUANTIFIERS:

- $\forall x \in A \iff$ "**for all** x in set A" $\iff$ "**for every** x in set A"
- $\exists x \in A \iff$ "**there exists** at least one element x in set A" $\iff$ "there exists an element x in set A"
- e.g. $\left(\forall w \in \mathbb{R}, \exists y \in \mathbb{Q} \text{ s.t. } w + y < 0\right) \iff$ "For every real w, there exists a rational y such that their sum is negative."