LINEAR SYSTEMS ($A\mathbf{x} = \mathbf{b}$): INTRO, INTERPRETATION [LARSON 1.1]

- **SCALAR:** A **scalar** is a **real #**: $0, 1, -5, 9/7, 2.4, \sqrt[3]{7}, 6-\sqrt{2}, \log 7, \pi, e, \sin(\pi/7), \ldots$

  Complex numbers will never be considered in this course: $3i$, $5-2i$ where $i := \sqrt{-1}$

- **LINEAR SYSTEMS:**

  A $m \times n$ **linear system in** $x_1, x_2, \ldots, x_n$ has the following form:
  $$\begin{cases} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{cases}$$

  where scalars $a_{11}, \ldots, a_{1n}, a_{21}, \ldots, a_{2n}, \ldots, a_{m1}, \ldots, a_{mn} \in \mathbb{R}$

  and the RHS scalars $b_1, \ldots, b_m \in \mathbb{R}$. [RHS ≡ "**R**ight-**H**and **S**ide"]

  The **unknown variables** are $x_1, \ldots, x_n$, **and appear only to the first power**.

  i.e., a $m \times n$ linear system has m linear equations & n unknown variables.

- **NON-LINEAR SYSTEMS:**

  A system of equations which is not linear is called a **nonlinear system**.

  Occasionally, it's possible to convert a nonlinear system into a linear system by a **change of variables (CV)**:

  CV: Let $u = 1/x$ and $v = \sqrt{y}$. Then: $\begin{cases} 4/x & - & 3\sqrt{y} & = & 1 \\ -1/x & + & \pi\sqrt{y} & = & 0 \end{cases} \overset{CV}{\iff} \begin{cases} 4u & - & 3v & = & 1 \\ -u & + & \pi v & = & 0 \end{cases}$

- **UNDERDETERMINED, OVERDETERMINED, SQUARE LINEAR SYSTEMS:**

  A $m \times n$ linear system is **underdetermined** $\iff$ there's more unknowns than equations ($m < n$)
  A $m \times n$ linear system is **overdetermined** $\iff$ there's more equations than unknowns ($m > n$)
  A $m \times n$ linear system is **square** $\iff$ there's as many equations as unknowns ($m = n$)

- **QUALITATIVE SOLUTION POSSIBLITILES FOR LINEAR SYSTEMS:**

  For a $m \times n$ linear system, exactly one of the following is true:

  - The linear system has a **unique solution** (i.e. one and only one solution)
  - The linear system has **infinitely many solutions**
  - The linear system has **no solution**

- **CONSISTENT & INCONSISTENT LINEAR SYSTEMS:**

  A $m \times n$ linear system is **inconsistent** $\iff$ it has **no solution**. Otherwise, the linear system is called **consistent**.

- **MATRICES, ROW VECTORS, COLUMN VECTORS:**

  - A $m \times n$ **matrix** is an array of scalars arranged in m rows & n columns. e.g. $\begin{bmatrix} 1 & 4 & 0 \\ -1 & \pi & (8-3\sqrt{5}) \end{bmatrix}$
  - A **square matrix** is a matrix with as many rows as columns ($m = n$). e.g. $\begin{bmatrix} \sqrt[3]{2} & 3/4 \\ 1 & 0 \end{bmatrix}$
  - A n**-wide row vector** is a $1 \times n$ matrix (i.e. only one row) e.g. $\begin{bmatrix} -3 & 7 & 0 & \sqrt{3} \end{bmatrix}$
  - A m**-wide column vector** is a $m \times 1$ matrix (i.e. only one column) e.g. $\begin{bmatrix} 7/5 \\ 1 \end{bmatrix}$

- **WRITING LINEAR SYSTEM IN MATRIX-VECTOR FORM:**

  The above prototype $m \times n$ linear system can be written compactly as $A\mathbf{x} = \mathbf{b}$,

  where A is $m \times n$ **matrix**, $\mathbf{x}$ is a n-wide **column vector**, and $\mathbf{b}$ is a m-wide **column vector**.

- **WRITING LINEAR SYSTEM AS AUGMENTED MATRIX:**

  A linear system $A\mathbf{x} = \mathbf{b}$ can be written compactly as $[\, A \mid \mathbf{b} \,]$: e.g. $\begin{cases} x & - & y & = & 5 \\ 2x & + & y & = & \sqrt[3]{9} \end{cases} \iff \left[\begin{array}{cc|c} 1 & -1 & 5 \\ 2 & 1 & \sqrt[3]{9} \end{array}\right]$