EX 1.2.3: Using Gauss-Jordan elimination, solve the linear system: $\begin{cases} 4x + 4y = 4 \\ 3x + y = 1 \\ 2x + y = 1 \\ 4x + 3y = 3 \end{cases}$

$$\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right] = \left[\begin{array}{cc|c} 4 & 4 & 4 \\ 3 & 1 & 1 \\ 2 & 1 & 1 \\ 4 & 3 & 3 \end{array}\right] \xrightarrow{(\frac{1}{4})R_1 \to R_1} \left[\begin{array}{cc|c} \boxed{1} & 1 & 1 \\ 3 & 1 & 1 \\ 2 & 1 & 1 \\ 4 & 3 & 3 \end{array}\right] \xrightarrow[(-4)R_1+R_4\to R_4]{\substack{(-3)R_1+R_2\to R_2 \\ (-2)R_1+R_3\to R_3}} \left[\begin{array}{cc|c} \boxed{1} & 1 & 1 \\ 0 & -2 & -2 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{array}\right]$$

$$\xrightarrow{(-\frac{1}{2})R_2 \to R_2} \left[\begin{array}{cc|c} \boxed{1} & 1 & 1 \\ 0 & \boxed{1} & 1 \\ 0 & -1 & -1 \\ 0 & -1 & -1 \end{array}\right] \xrightarrow[R_2+R_4\to R_4]{R_2+R_3\to R_3} \left[\begin{array}{cc|c} \boxed{1} & 1 & 1 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{(-1)R_2+R_1\to R_1} \left[\begin{array}{cc|c} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} \text{RREF}(A) & \widetilde{\mathbf{b}} \end{array}\right]$$

Notice that rows 3 & 4 of RREF(A) are TAUTOLOGIES, since they each translate to the equation $0 = 0$.

Remember, nothing special is done when encountering TAUTOLOGIES - just think "Oh, that's nice." and carry on.

Identify the **free** & **fixed** variables:

Every column of RREF(A) has pivot $\implies$ all unknowns x, y are **fixed variables** (there are no **free variables**)

Qualitatively describe the solution(s) to the linear system:

Since there's no CONTRADICTION's & **every unknown is a fixed variable**, the linear system has a **unique solution**.

Find the solution(s) to the linear system:

Translate $\left[\begin{array}{c|c} \text{RREF}(A) & \widetilde{\mathbf{b}} \end{array}\right]$ into an equivalent linear system:

$$\left[\begin{array}{cc|c} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \iff \begin{cases} x = 0 \\ y = 1 \\ 0 = 0 \\ 0 = 0 \end{cases} \iff \begin{cases} x = 0 \\ y = 1 \end{cases}$$

$\therefore$ (TUPLE FORM) $\boxed{(x, y) = (0, 1)}$ OR (COLUMN VECTOR FORM) $\boxed{\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$

EX 1.2.4: Using Gauss-Jordan elimination, solve the linear system: $\begin{cases} -x_1 + 3x_2 = -1 \\ 3x_1 - 4x_2 = -1 \\ -3x_1 + 2x_2 = 4 \\ -9x_1 + 6x_2 = -3 \end{cases}$

$$\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right] = \left[\begin{array}{cc|c} -1 & 3 & -1 \\ 3 & -4 & -1 \\ -3 & 2 & 4 \\ -9 & 6 & -3 \end{array}\right] \xrightarrow{(-1)R_1 \to R_1} \left[\begin{array}{cc|c} \boxed{1} & -3 & 1 \\ 3 & -4 & -1 \\ -3 & 2 & 4 \\ -9 & 6 & -3 \end{array}\right] \xrightarrow[(9)R_1+R_4\to R_4]{\substack{(-3)R_1+R_2\to R_2 \\ (3)R_1+R_3\to R_3}} \left[\begin{array}{cc|c} \boxed{1} & -3 & 1 \\ 0 & 5 & -4 \\ 0 & -7 & 7 \\ 0 & -21 & 6 \end{array}\right]$$

$$\xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{cc|c} \boxed{1} & -3 & 1 \\ 0 & -7 & 7 \\ 0 & 5 & -4 \\ 0 & -21 & 6 \end{array}\right] \xrightarrow{(-\frac{1}{7})R_2 \to R_2} \left[\begin{array}{cc|c} \boxed{1} & -3 & 1 \\ 0 & \boxed{1} & -1 \\ 0 & 5 & -4 \\ 0 & -21 & 6 \end{array}\right] \xrightarrow[(21)R_2+R_4\to R_4]{(-5)R_2+R_3\to R_3} \left[\begin{array}{cc|c} \boxed{1} & -3 & 1 \\ 0 & \boxed{1} & -1 \\ 0 & 0 & 1 \\ 0 & 0 & -15 \end{array}\right] \begin{array}{l} \\ \\ \leftarrow \text{CONTRADICTION!} \\ \leftarrow \text{CONTRADICTION!} \end{array}$$

$\therefore$ Since there's at least one CONTRADICTION, the linear system has $\boxed{\textbf{no solution}}$

EX 1.2.5: Using Gauss-Jordan elimination, solve the linear system: $\begin{cases} -x_1 + 2x_2 = -1 \\ x_1 - 2x_2 = 1 \\ 3x_1 - 6x_2 = 3 \\ -4x_1 + 8x_2 = -4 \end{cases}$

$$\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right] = \left[\begin{array}{cc|c} -1 & 2 & -1 \\ 1 & -2 & 1 \\ 3 & -6 & 3 \\ -4 & 8 & -4 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} \boxed{1} & -2 & 1 \\ -1 & 2 & -1 \\ 3 & -6 & 3 \\ -4 & 8 & -4 \end{array}\right] \xrightarrow[4R_1+R_4 \to R_4]{\substack{R_1+R_2 \to R_2 \\ (-3)R_1+R_3 \to R_3}} \left[\begin{array}{cc|c} \boxed{1} & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{c|c} \text{RREF}(A) & \widetilde{\mathbf{b}} \end{array}\right]$$

Notice that rows 2,3,4 of RREF(A) are TAUTOLOGIES, since they each translate to the equation $0 = 0$.

Remember, nothing special is done when encountering a TAUTOLOGY - just think "Oh, that's nice." and carry on.

Identify the **free** & **fixed** variables:

Column 2 of RREF(A) has **no pivot** $\implies$ Unknown x_2 is a **free variable**.
Column 1 of RREF(A) has a pivot $\implies$ Unknown x_1 is a **fixed variable**.

Qualitatively describe the solution(s) to the linear system:

Since there's no CONTRADICTIONS & there's at least one **free variable**, there are **infinitely many solutions**.

Since there's **2 unknowns** (x_1, x_2) & **one free variable** (x_2), all solutions lie on a **line** in **2**-dimensional space.

Find the solution(s) to the linear system:

Assign a **unique parameter** to each **free variable**: Let $x_2 = t \in \mathbb{R}$

Express all **fixed variables** in terms of the **parameters** by translating $\left[\begin{array}{c|c} \text{RREF}(A) & \widetilde{\mathbf{b}} \end{array}\right]$ into equivalent linear system:

$$\left[\begin{array}{cc|c} \boxed{1} & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] \iff \begin{cases} x_1 - 2x_2 = 1 \\ 0 = 0 \\ 0 = 0 \\ 0 = 0 \end{cases} \iff x_1 - 2t = 1 \iff x_1 = 1 + 2t$$

$\therefore$ (TUPLE FORM) $\boxed{(x_1, x_2) = (1 + 2t, t)}$

OR

$\therefore$ (COLUMN VECTOR FORM) $\boxed{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1+2t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \end{bmatrix}}$

These two forms of the solution are shown because each WeBWorK problem will demand one form or the other.

On exams, either form is fine with me.

EX 1.2.6: Using Gauss-Jordan elimination, solve the linear system: $\begin{cases} & & & & -3x_4 & = & 2 \\ -2x_1 & & - & 4x_3 & + & 4x_4 & = & 1 \\ 2x_1 & - & x_2 & + & 4x_3 & - & 3x_4 & = & -1 \end{cases}$

IMPORTANT: Since the $(1,1)$-entry of $\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right]$ is **zero**, **perform a SWAP first!**

$$\left[\begin{array}{c|c} A & \mathbf{b} \end{array}\right] = \left[\begin{array}{cccc|c} 0 & 0 & 0 & -3 & 2 \\ -2 & 0 & -4 & 4 & 1 \\ 2 & -1 & 4 & -3 & -1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{cccc|c} 2 & -1 & 4 & -3 & -1 \\ -2 & 0 & -4 & 4 & 1 \\ 0 & 0 & 0 & -3 & 2 \end{array}\right]$$

$$\xrightarrow{R_1+R_2 \to R_2} \left[\begin{array}{cccc|c} 2 & -1 & 4 & -3 & -1 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 & 2 \end{array}\right] \xrightarrow[\left(-\frac{1}{3}\right)R_3 \to R_3]{\left(\frac{1}{2}\right)R_1 \to R_1 \quad (-1)R_2 \to R_2} \left[\begin{array}{cccc|c} \boxed{1} & -1/2 & 2 & -3/2 & -1/2 \\ 0 & \boxed{1} & 0 & -1 & 0 \\ 0 & 0 & 0 & \boxed{1} & -2/3 \end{array}\right]$$

$$\xrightarrow[\left(\frac{3}{2}\right)R_3+R_1 \to R_1]{R_3+R_2 \to R_2} \left[\begin{array}{cccc|c} \boxed{1} & -1/2 & 2 & 0 & -3/2 \\ 0 & \boxed{1} & 0 & 0 & -2/3 \\ 0 & 0 & 0 & \boxed{1} & -2/3 \end{array}\right] \xrightarrow{\left(\frac{1}{2}\right)R_2+R_1 \to R_1} \left[\begin{array}{cccc|c} \boxed{1} & 0 & 2 & 0 & -11/6 \\ 0 & \boxed{1} & 0 & 0 & -2/3 \\ 0 & 0 & 0 & \boxed{1} & -2/3 \end{array}\right] = \left[\begin{array}{c|c} \text{RREF}(A) & \widetilde{\mathbf{b}} \end{array}\right]$$

Identify the **free** & **fixed** variables:

Column 3 of RREF(A) has **no pivot** $\implies$ Unknown x_3 is a **free variable**.
Columns 1,2,4 of RREF(A) each have a pivot $\implies$ Unknowns x_1, x_2, x_4 are **fixed variables**.

Qualitatively describe the solution(s) to the linear system:

Since there's no CONTRADICTIONS & there's at least one **free variable**, there are **infinitely many solutions**.

Since there's **4 unknowns** (x_1, x_2, x_3, x_4) & **one free variable** (x_3), all solutions lie on a **line** in **4**-dimensional space.

Find the solution(s) to the linear system:

Assign a **unique parameter** to each **free variable**: Let $x_3 = t \in \mathbb{R}$

Express all **fixed variables** in terms of the **parameters** by translating $\left[\begin{array}{c|c} \text{RREF}(A) & \widetilde{\mathbf{b}} \end{array}\right]$ into equivalent linear system:

$$\left[\begin{array}{cccc|c} \boxed{1} & 0 & 2 & 0 & -11/6 \\ 0 & \boxed{1} & 0 & 0 & -2/3 \\ 0 & 0 & 0 & \boxed{1} & -2/3 \end{array}\right] \iff \begin{cases} x_1 + 2x_3 = -11/6 \\ x_2 = -2/3 \\ x_4 = -2/3 \end{cases} \iff \begin{cases} x_1 + 2t = -11/6 \\ x_2 = -2/3 \\ x_4 = -2/3 \end{cases}$$

$\therefore$ (TUPLE FORM) $\boxed{(x_1, x_2, x_3, x_4) = \left(-\frac{11}{6} - 2t, -\frac{2}{3}, t, -\frac{2}{3}\right)}$

— OR —

$\therefore$ (COLUMN VECTOR FORM) $\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -\frac{11}{6} - 2t \\ -\frac{2}{3} \\ t \\ -\frac{2}{3} \end{bmatrix} = \begin{bmatrix} -\frac{11}{6} \\ -\frac{2}{3} \\ 0 \\ -\frac{2}{3} \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}}$

These two forms of the solution are shown because each WeBWorK problem will demand one form or the other.

On exams, either form is fine with me.

EX 1.2.7: Using Gauss-Jordan elimination, solve the linear system: $\begin{cases} x_1 - x_2 = 0 \\ -2x_2 - 2x_3 = 0 \\ -x_2 - x_3 = 0 \end{cases}$

$$\left[\, A \mid \mathbf{b} \,\right] = \left[\begin{array}{ccc|c} \boxed{1} & -1 & 0 & 0 \\ -2 & 0 & -2 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right] \xrightarrow{2R_1+R_2\to R_2} \left[\begin{array}{ccc|c} \boxed{1} & -1 & 0 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right] \xrightarrow{(-\frac{1}{2})R_2\to R_2} \left[\begin{array}{ccc|c} \boxed{1} & -1 & 0 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & -1 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2+R_3\to R_3} \left[\begin{array}{ccc|c} \boxed{1} & -1 & 0 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_2+R_1\to R_1} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[\, \text{RREF}(A) \mid \widetilde{\mathbf{b}} \,\right]$$

Notice that row 3 of RREF(A) is a TAUTOLOGY, since it translates to the equation $0 = 0$.

Remember, nothing special is done when encountering a TAUTOLOGY - just think "Oh, that's nice." and carry on.

Identify the **free** & **fixed** variables:

Column 3 of RREF(A) has **no pivot** $\implies$ Unknown x_3 is a **free variable**.

Columns 1,2 of RREF(A) each have a pivot $\implies$ Unknowns x_1, x_2 are **fixed variables**.

Qualitatively describe the solution(s) to the linear system:

Since there's no CONTRADICTIONS & there's at least one **free variable**, there are **infinitely many solutions**.

Since there's **3 unknowns** (x_1, x_2, x_3) & **one** **free variable** (x_3), all solutions lie on a **line** in **3**-dimensional space.

Find the solution(s) to the linear system:

Assign a **unique parameter** to each **free variable**: Let $x_3 = t \in \mathbb{R}$

Express all **fixed variables** in terms of the **parameters** by translating $\left[\, \text{RREF}(A) \mid \widetilde{\mathbf{b}} \,\right]$ into equivalent linear system:

$$\left[\begin{array}{ccc|c} \boxed{1} & 0 & 1 & 0 \\ 0 & \boxed{1} & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \iff \begin{cases} x_1 + x_3 = 0 \\ x_2 + x_3 = 0 \\ 0 = 0 \end{cases} \iff \begin{cases} x_1 + t = 0 \\ x_2 + t = 0 \end{cases} \iff \begin{cases} x_1 = -t \\ x_2 = -t \end{cases}$$

$\therefore$ (TUPLE FORM) $\boxed{(x_1, x_2, x_3) = (-t, -t, t)}$

OR

$\therefore$ (COLUMN VECTOR FORM) $\boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}}$

These two forms of the solution are shown because each WeBWorK problem will demand one form or the other.

On exams, either form is fine with me.

EX 1.2.8: Using Gauss-Jordan elimination, solve the linear system: $\begin{cases} x_1 + 2x_2 - 3x_3 = 0 \\ 2x_1 + 4x_2 - 6x_3 = 0 \\ -3x_1 - 6x_2 + 9x_3 = 0 \end{cases}$

$$\left[\, A \mid \mathbf{b} \,\right] = \left[\begin{array}{ccc|c} \boxed{1} & 2 & -3 & 0 \\ 2 & 4 & -6 & 0 \\ -3 & -6 & 9 & 0 \end{array}\right] \xrightarrow[3R_1+R_3\to R_3]{(-2)R_1+R_2\to R_2} \left[\begin{array}{ccc|c} \boxed{1} & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[\, \text{RREF}(A) \mid \widetilde{\mathbf{b}} \,\right]$$

Notice that rows 2 & 3 of RREF(A) are TAUTOLOGIES, since they each translate to the equation $0 = 0$.

Remember, nothing special is done when encountering TAUTOLOGIES - just think "Oh, that's nice." and carry on.

Identify the **free** & **fixed** variables:

Columns 2,3 of RREF(A) each have **no pivot** $\implies$ Unknowns x_2, x_3 are **free variables**.

Columns 1 of RREF(A) has a pivot $\implies$ Unknown x_1 is a **fixed variable**.

Qualitatively describe the solution(s) to the linear system:

Since there's no CONTRADICTIONS & there's at least one **free variable**, there are **infinitely many solutions**.

Since there's **3 unknowns** (x_1, x_2, x_3) & **two free variables** (x_2, x_3), all solutions lie on a **plane** in **3**-dimensional space.

Find the solution(s) to the linear system:

Assign a **unique parameter** to each **free variable**: Let $x_2 = s \in \mathbb{R}$
Let $x_3 = t \in \mathbb{R}$

Express all **fixed variables** in terms of the **parameters** by translating $\left[\, \text{RREF}(A) \mid \widetilde{\mathbf{b}} \,\right]$ into equivalent linear system:

$$\left[\begin{array}{ccc|c} \boxed{1} & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \iff \begin{cases} x_1 + 2x_2 - 3x_3 = 0 \\ 0 = 0 \\ 0 = 0 \end{cases} \iff x_1 + 2s - 3t = 0 \iff x_1 = -2s + 3t$$

$$\therefore \boxed{(x_1, x_2, x_3) = (-2s + 3t, s, t)} \text{ OR } \boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2s + 3t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}}$$