SOLVING $A\mathbf{x} = \mathbf{b}$: GAUSS-JORDAN ELIMINATION [LARSON 1.2]

- **EQUIVALENT LINEAR SYSTEMS:**
  - Two $m \times n$ linear systems are **equivalent** $\iff$ both systems have the **exact same solution sets**.
  - When solving a linear system $A\mathbf{x} = \mathbf{b}$, one should rewrite the system into a **simpler equivalent system**.
    This can achieved using **elementary row operations** applied to the corresponding **augmented matrix** $[\ A \mid \mathbf{b}\ ]$.

- **ELEMENTARY ROW OPERATIONS:**
  - ⋆ (SWAP) $[R_i \leftrightarrow R_j]$ Swap row i & row j:
    e.g. Swap row 2 & row 3: $\left[\begin{array}{rr|r} 3 & 4 & 0 \\ -1 & 0 & 1 \\ 2 & 7 & 9 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{rr|r} 3 & 4 & 0 \\ 2 & 7 & 9 \\ -1 & 0 & 1 \end{array}\right]$
  - ⋆ (SCALE) $[\alpha R_j \to R_j]$ Multiply row j by a non-zero scalar α:
    e.g. Multiply row 1 by -2: $\left[\begin{array}{rr|r} 3 & 4 & 0 \\ -1 & 0 & 1 \\ 2 & 7 & 9 \end{array}\right] \xrightarrow{(-2)R_1 \to R_1} \left[\begin{array}{rr|r} (-2)(3) & (-2)(4) & (-2)(0) \\ -1 & 0 & 1 \\ 2 & 7 & 9 \end{array}\right] = \left[\begin{array}{rr|r} -6 & -8 & 0 \\ -1 & 0 & 1 \\ 2 & 7 & 9 \end{array}\right]$
  - ⋆ (COMBINE) $[\alpha R_i + R_j \to R_j]$ Add scalar multiple α of row i to row j:
    e.g. Add (3×row 1) to row 3: $\left[\begin{array}{rr|r} 3 & 4 & 0 \\ -1 & 0 & 1 \\ 2 & 7 & 9 \end{array}\right] \xrightarrow{3R_1 + R_3 \to R_3} \left[\begin{array}{rr|r} 3 & 4 & 0 \\ -1 & 0 & 1 \\ 3(3)+2 & 3(4)+7 & 3(0)+9 \end{array}\right] = \left[\begin{array}{rr|r} 3 & 4 & 0 \\ -1 & 0 & 1 \\ 11 & 19 & 9 \end{array}\right]$

- **REDUCED ROW-ECHELON FORM (RREF) OF A MATRIX:**
  - An matrix is in **reduced row-echelon form (RREF)** if the following are all true:
    - ∗ Any rows consisting entirely of zeros occur below all non-zero rows.
    - ∗ For each non-zero row, the **first (leftmost) non-zero entry** is **1** (and is called a **pivot** or **leading one**.)
    - ∗ For two successive non-zero rows, the pivot in the higher row is farther to the left than the pivot in lower row.
    - ∗ Every column that has a pivot has zeros in every entry above & below its pivot.
    - ∗ For **linear systems**, the $(1,1)$-entry must be a **pivot**. ["(i,j)-entry" means "i^{th} row, j^{th} column of matrix"]
  - The RREF of an augmented matrix seemingly may have a "pivot" in the last column, but it's not really a pivot!
    However, still zero out entries above & below such a "pivot" in the last column.
  - Examples of augmented matrices in RREF (pivots are boxed):

$$\left[\begin{array}{rrr|r} \boxed{1} & 0 & 0 & 3 \\ 0 & \boxed{1} & 0 & 5 \\ 0 & 0 & \boxed{1} & 1 \end{array}\right], \left[\begin{array}{rrr|r} \boxed{1} & 6 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right], \left[\begin{array}{rrr|r} \boxed{1} & 6 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right], \left[\begin{array}{rr|r} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 1 \end{array}\right], \left[\begin{array}{rr|r} \boxed{1} & -3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right], \left[\begin{array}{rr|r} \boxed{1} & -3 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

- **SOLVING $m \times n$ LINEAR SYSTEM $A\mathbf{x} = \mathbf{b}$ USING GAUSS-JORDAN ELIMINATION (PROCEDURE):**
  (1) Form **augmented matrix** $[\ A \mid \mathbf{b}\ ]$
  (2) SWAP/SCALE/COMBINE $[\ A \mid \mathbf{b}\ ]$ as needed to zero-out entries **below pivots, left-to-right**
  (3) SWAP/SCALE/COMBINE $[\ A \mid \mathbf{b}\ ]$ as needed to zero-out entries **above pivots, right-to-left**
  (4) Translate augmented matrix $[\ \mathrm{RREF}(A) \mid \widetilde{\mathbf{b}}\ ]$ into corresponding equivalent linear system:
      If any rows of $[\ \mathrm{RREF}(A) \mid \widetilde{\mathbf{b}}\ ]$ translate to a **CONTRADICTION** (e.g. $0 = 1$) then system has **no solution**
      If any rows of $[\ \mathrm{RREF}(A) \mid \widetilde{\mathbf{b}}\ ]$ translate to a **TAUTOLOGY** (e.g. $0 = 0$), then proceed as usual to STEP (5)
  (5) Identify all **fixed variables** & **free variables**:
      Each column of $\mathrm{RREF}(A)$ that contains **a pivot** corresponds to a **fixed variable**
      Each column of $\mathrm{RREF}(A)$ that contains **no pivot** corresponds to a **free variable**
  (6) Assign each **free variable** a unique **parameter**
  (7) Express each **fixed variable** in terms of the **parameters**
  (8) Write out **solution** in either **tuple form** or **column vector form**

SOLVING $A\mathbf{x} = \mathbf{b}$: GAUSS-JORDAN ELIMINATION [LARSON 1.2]

- **TUPLE FORM OF A SOLUTION:** $(x_1, x_2) = (3, -5)$, $(x_1, x_2, x_3) = (3 - 6s - 2t, s, t)$
- **COLUMN VECTOR FORM OF A SOLUTION:**

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \end{bmatrix}, \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3-6s-2t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} -6 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

- **GAUSS-JORDAN ELIMINATION (3×2 PROTOTYPE POSSIBILITIES):**

$*$ indicates possibly non-zero entries   Pivots are boxed:

$$[\ A \mid \mathbf{b}\ ] = \left[\begin{array}{cc|c} * & * & * \\ * & * & * \\ * & * & * \end{array}\right] \xrightarrow{Gauss-Jordan} \left[\begin{array}{cc|c} \boxed{1} & 0 & * \\ 0 & \boxed{1} & * \\ 0 & 0 & 0 \end{array}\right] = \left[\ \mathrm{RREF}(A) \ \middle|\ \widetilde{\mathbf{b}}\ \right]$$

$$[\ A \mid \mathbf{b}\ ] = \left[\begin{array}{cc|c} * & * & * \\ * & * & * \\ * & * & * \end{array}\right] \xrightarrow{Gauss-Jordan} \left[\begin{array}{cc|c} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & 1 \end{array}\right] = \left[\ \mathrm{RREF}(A) \ \middle|\ \widetilde{\mathbf{b}}\ \right]$$

$$[\ A \mid \mathbf{b}\ ] = \left[\begin{array}{cc|c} * & * & * \\ * & * & * \\ * & * & * \end{array}\right] \xrightarrow{Gauss-Jordan} \left[\begin{array}{cc|c} \boxed{1} & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right] = \left[\ \mathrm{RREF}(A) \ \middle|\ \widetilde{\mathbf{b}}\ \right]$$

$$[\ A \mid \mathbf{b}\ ] = \left[\begin{array}{cc|c} * & * & * \\ * & * & * \\ * & * & * \end{array}\right] \xrightarrow{Gauss-Jordan} \left[\begin{array}{cc|c} \boxed{1} & * & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right] = \left[\ \mathrm{RREF}(A) \ \middle|\ \widetilde{\mathbf{b}}\ \right]$$

- **[PROTIP] DELAY THE ONSLAUGHT OF FRACTIONS (PART 1):**

Sometimes an augmented matrix may have **fractions** in some entries:

$$\left[\begin{array}{cc|c} 3 & \mathbf{1/2} & \mathbf{1/5} \\ \mathbf{1/3} & 2 & 4 \end{array}\right] \xrightarrow{(\frac{1}{3})R_1 \to R_1} \left[\begin{array}{cc|c} \boxed{1} & \mathbf{1/6} & \mathbf{1/15} \\ \mathbf{1/3} & 2 & 4 \end{array}\right]$$

This will cause Gauss-Jordan to involve tedious fraction arithmetic!

To avoid dealing with fractions (at least for a few steps), **SCALE** each row with fractions by its **common denominator**:

$$\left[\begin{array}{cc|c} 3 & \mathbf{1/2} & \mathbf{1/5} \\ \mathbf{1/3} & 2 & 4 \end{array}\right] \xrightarrow[3R_2 \to R_2]{10R_1 \to R_1} \left[\begin{array}{cc|c} 30 & \mathbf{5} & \mathbf{2} \\ 1 & 6 & 12 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{cc|c} \boxed{1} & \mathbf{6} & \mathbf{12} \\ \mathbf{30} & 5 & 2 \end{array}\right]$$

- **[PROTIP] DELAY THE ONSLAUGHT OF FRACTIONS (PART 2):**

Sometimes a SCALE to create a pivot may cause **fractions** in other entries:

$$\left[\begin{array}{cc|c} 3 & 4 & 8 \\ 2 & 3 & 0 \\ 5 & 6 & 3 \end{array}\right] \xrightarrow{(\frac{1}{3})R_1 \to R_1} \left[\begin{array}{cc|c} \boxed{1} & \mathbf{4/3} & \mathbf{8/3} \\ 2 & 3 & 0 \\ 5 & 6 & 3 \end{array}\right]$$

This will cause later Gauss-Jordan steps to involve tedious fraction arithmetic!

To avoid dealing with fractions (at least for a few steps), zero-out each entry below a would-be pivot by **SCALE**-ing **each pair of rows** such that the two entries are **identical**:

$$\left[\begin{array}{cc|c} \mathbf{3} & 4 & 8 \\ \mathbf{2} & 3 & 0 \\ 5 & 6 & 3 \end{array}\right] \xrightarrow[3R_2 \to R_2]{2R_1 \to R_1} \left[\begin{array}{cc|c} \mathbf{6} & 8 & 16 \\ \mathbf{6} & 9 & 0 \\ 5 & 6 & 3 \end{array}\right] \xrightarrow{(-1)R_1 + R_2 \to R_2} \left[\begin{array}{cc|c} 6 & 8 & 16 \\ \mathbf{0} & 1 & -16 \\ 5 & 6 & 3 \end{array}\right] \xrightarrow{(\frac{1}{2})R_1 \to R_1}$$

$$\left[\begin{array}{cc|c} \mathbf{3} & 4 & 8 \\ \mathbf{0} & 1 & -16 \\ \mathbf{5} & 6 & 3 \end{array}\right] \xrightarrow[3R_3 \to R_3]{5R_1 \to R_1} \left[\begin{array}{cc|c} \mathbf{15} & 20 & 40 \\ \mathbf{0} & 1 & -16 \\ \mathbf{15} & 18 & 9 \end{array}\right] \xrightarrow{(-1)R_1 + R_3 \to R_3} \left[\begin{array}{cc|c} 15 & 20 & 40 \\ \mathbf{0} & 1 & -16 \\ \mathbf{0} & -2 & -31 \end{array}\right]$$

$$\xrightarrow{(\frac{1}{15})R_1 \to R_1} \left[\begin{array}{cc|c} \boxed{1} & \mathbf{4/3} & \mathbf{8/3} \\ \mathbf{0} & 1 & -16 \\ \mathbf{0} & -2 & -31 \end{array}\right]$$

(Again, fractions may be inevitable, but at least they can be delayed. NOTE: entries may become **quite large**!)

EX 1.2.1: Using Gauss-Jordan elimination, solve the linear system: $\begin{cases} x_1 - 2x_2 - 2x_3 + 2x_4 = 0 \\ -x_1 - 2x_2 + x_4 = 1 \\ 2x_1 - 4x_2 - 4x_3 + 4x_4 = 3 \end{cases}$

EX 1.2.2: Using Gauss-Jordan elimination, solve the linear system: $\begin{cases} x + y + 2z + 2w = 3 \\ x + 2y + z + 2w = 1 \\ -2x - 2y - 4z - 4w = -6 \end{cases}$

EX 1.2.3: Using Gauss-Jordan elimination, solve the linear system: $\begin{cases} 4x & + & 4y & = & 4 \\ 3x & + & y & = & 1 \\ 2x & + & y & = & 1 \\ 4x & + & 3y & = & 3 \end{cases}$

EX 1.2.4: Using Gauss-Jordan elimination, solve the linear system: $\begin{cases} -x_1 & + & 3x_2 & = & -1 \\ 3x_1 & - & 4x_2 & = & -1 \\ -3x_1 & + & 2x_2 & = & 4 \\ -9x_1 & + & 6x_2 & = & -3 \end{cases}$

EX 1.2.5: Using Gauss-Jordan elimination, solve the linear system:

$$\begin{cases} -x_1 + 2x_2 = -1 \\ x_1 - 2x_2 = 1 \\ 3x_1 - 6x_2 = 3 \\ -4x_1 + 8x_2 = -4 \end{cases}$$

EX 1.2.6: Using Gauss-Jordan elimination, solve the linear system:

$$\begin{cases} -3x_4 = 2 \\ -2x_1 - 4x_3 + 4x_4 = 1 \\ 2x_1 - x_2 + 4x_3 - 3x_4 = -1 \end{cases}$$

EX 1.2.7: Using Gauss-Jordan elimination, solve the linear system: $\begin{cases} x_1 - x_2 = 0 \\ -2x_2 - 2x_3 = 0 \\ - x_2 - x_3 = 0 \end{cases}$

EX 1.2.8: Using Gauss-Jordan elimination, solve the linear system: $\begin{cases} x_1 + 2x_2 - 3x_3 = 0 \\ 2x_1 + 4x_2 - 6x_3 = 0 \\ -3x_1 - 6x_2 + 9x_3 = 0 \end{cases}$