CONSTRUCTING $A\mathbf{x} = \mathbf{b}$: CURVE INTERPOLATION [LARSON 1.3]

EX 1.3.1: Find a quadratic polynomial $p(x) = c_0 + c_1 x + c_2 x^2$ such that it contains the points $(-1, 4), (0, -2), (3, 5)$.

In other words, p must satisfy: $p(-1) = 4, \quad p(0) = -2, \quad p(3) = 5$.

1^{st}, setup the linear system for $p(x)$ which satisfies the three conditions:

$$\begin{cases} p(-1) = 4 \\ p(0) = -2 \\ p(3) = 5 \end{cases} \implies \begin{cases} c_0 + c_1(-1) + c_2(-1)^2 = 4 \\ c_0 + c_1(0) + c_2(0)^2 = -2 \\ c_0 + c_1(3) + c_2(3)^2 = 5 \end{cases} \implies \underbrace{\begin{cases} c_0 - c_1 + c_2 = 4 \\ c_0 = -2 \\ c_0 + 3c_1 + 9c_2 = 5 \end{cases}}_{A\mathbf{x}=\mathbf{b} \quad \text{(or some books write } X\mathbf{c}=\mathbf{y})}$$

2^{nd}, form augmented matrix $[\ A \mid \mathbf{b}\ ]$ (or $[\ X \mid \mathbf{y}\ ]$) and perform Gauss-Jordan Elimination:

$$\underbrace{[\ A \mid \mathbf{b}\ ]}_{=[\ X \mid \mathbf{y}\ ]} = \left[\begin{array}{ccc|c} 1 & -1 & 1 & 4 \\ 1 & 0 & 0 & -2 \\ 1 & 3 & 9 & 5 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & -2 \\ 1 & -1 & 1 & 4 \\ 1 & 3 & 9 & 5 \end{array}\right] \xrightarrow[(-1)R_1+R_3 \to R_3]{(-1)R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & -2 \\ 0 & -1 & 1 & 6 \\ 0 & 3 & 9 & 7 \end{array}\right]$$

$$\xrightarrow{(-1)R_2 \to R_2} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & -2 \\ 0 & \boxed{1} & -1 & -6 \\ 0 & 3 & 9 & 7 \end{array}\right] \xrightarrow{(-3)R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & -2 \\ 0 & \boxed{1} & -1 & -6 \\ 0 & 0 & 12 & 25 \end{array}\right] \xrightarrow{(\frac{1}{12})R_3 \to R_3} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & -2 \\ 0 & \boxed{1} & -1 & -6 \\ 0 & 0 & \boxed{1} & 25/12 \end{array}\right]$$

$$\xrightarrow{R_3+R_2 \to R_2} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & -2 \\ 0 & \boxed{1} & 0 & -47/12 \\ 0 & 0 & \boxed{1} & 25/12 \end{array}\right] = [\ \text{RREF}(A) \mid \widetilde{\mathbf{b}}\ ] \quad (\text{ or } [\ \text{RREF}(X) \mid \widetilde{\mathbf{y}}\ ]\ )$$

3^{rd}, interpret $[\ A \mid \mathbf{b}\ ]$ (or $[\ X \mid \mathbf{y}\ ]$) to determine interpolating polynomial $p(x)$:

Notice that each column of RREF(A) (or RREF(X)) has a **pivot**,
which means the solution to the linear system $\mathbf{x}$ (or $\mathbf{c}$) is **unique**,
which means the interpolating polynomial $p(x)$ is **unique**.

$$\therefore \underbrace{\mathbf{x}}_{=\mathbf{c}} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -2 \\ -47/12 \\ 25/12 \end{bmatrix} \implies \boxed{p(x) = -2 - \frac{47}{12}x + \frac{25}{12}x^2}$$

REMARK #1:

Notice that the polynomial is written in **increasing-degree order** $(c_0 + c_1 x + c_2 x^2)$ rather than the usual **decreasing-degree order** $(c_2 x^2 + c_1 x + c_0)$ as this **guarantees** that the **first column** of augmented matrix will **always be all one's**, which makes Gauss-Jordan elimination somewhat easier to perform.

REMARK #2: When performing interpolation, it's typical that at least some of the coefficients are **fractions**.

EX 1.3.2: Find a function $f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$ such that: $f(2) = 3, f'(2) = -1, f''(2) = 1, f'''(2) = 0.$

1^{st}, compute the required derivatives of $f(x)$:

$$\begin{aligned} f'(x) &= \tfrac{d}{dx}\left[c_0 + c_1 x + c_2 x^2 + c_3 x^3\right] &&= c_1 + 2c_2 x + 3c_3 x^2 \\ f''(x) &= \tfrac{d}{dx}\left[c_1 + 2c_2 x + 3c_3 x^2\right] &&= 2c_2 + 6c_3 x \\ f'''(x) &= \tfrac{d}{dx}\left[2c_2 + 6c_3 x\right] &&= 6c_3 \end{aligned}$$

2^{nd}, setup the linear system for $f(x)$ which satisfies the four conditions:

$$\begin{cases} f(2) = 3 \\ f'(2) = -1 \\ f''(2) = 1 \\ f'''(2) = 0 \end{cases} \Longrightarrow \begin{cases} c_0 + c_1(2) + c_2(2)^2 + c_3(2)^3 = 3 \\ c_1 + 2c_2(2) + 3c_3(2)^2 = -1 \\ 2c_2 + 6c_3(2) = 1 \\ 6c_3 = 0 \end{cases} \Longrightarrow \underbrace{\begin{cases} c_0 + 2c_1 + 4c_2 + 8c_3 = 3 \\ c_1 + 4c_2 + 12c_3 = -1 \\ 2c_2 + 12c_3 = 1 \\ 6c_3 = 0 \end{cases}}_{A\mathbf{x}=\mathbf{b} \quad \text{(or some books write } X\mathbf{c}=\mathbf{y})}$$

3^{rd}, form augmented matrix $[\ A \mid \mathbf{b}\ ]$ (or $[\ X \mid \mathbf{y}\ ]$) and perform Gauss-Jordan Elimination:

$$\underbrace{[\ A \mid \mathbf{b}\ ]}_{=[\ X \mid \mathbf{y}\ ]} = \left[\begin{array}{cccc|c} \boxed{1} & 2 & 4 & 8 & 3 \\ 0 & \boxed{1} & 4 & 12 & -1 \\ 0 & 0 & 2 & 12 & 1 \\ 0 & 0 & 0 & 6 & 0 \end{array}\right] \xrightarrow[(\frac{1}{6})R_4 \to R_4]{(\frac{1}{2})R_3 \to R_3} \left[\begin{array}{cccc|c} \boxed{1} & 2 & 4 & 8 & 3 \\ 0 & \boxed{1} & 4 & 12 & -1 \\ 0 & 0 & \boxed{1} & 6 & 1/2 \\ 0 & 0 & 0 & \boxed{1} & 0 \end{array}\right]$$

$$\xrightarrow[(-8)R_4 + R_1 \to R_1]{\substack{(-6)R_4 + R_3 \to R_3 \\ (-12)R_4 + R_2 \to R_2}} \left[\begin{array}{cccc|c} \boxed{1} & 2 & 4 & 0 & 3 \\ 0 & \boxed{1} & 4 & 0 & -1 \\ 0 & 0 & \boxed{1} & 0 & 1/2 \\ 0 & 0 & 0 & \boxed{1} & 0 \end{array}\right] \xrightarrow[(-4)R_3 + R_1 \to R_1]{(-4)R_3 + R_2 \to R_2} \left[\begin{array}{cccc|c} \boxed{1} & 2 & 0 & 0 & 1 \\ 0 & \boxed{1} & 0 & 0 & -3 \\ 0 & 0 & \boxed{1} & 0 & 1/2 \\ 0 & 0 & 0 & \boxed{1} & 0 \end{array}\right]$$

$$\xrightarrow{(-2)R_2 + R_1 \to R_1} \left[\begin{array}{cccc|c} \boxed{1} & 0 & 0 & 0 & 7 \\ 0 & \boxed{1} & 0 & 0 & -3 \\ 0 & 0 & \boxed{1} & 0 & 1/2 \\ 0 & 0 & 0 & \boxed{1} & 0 \end{array}\right] = [\ \mathrm{RREF}(A) \mid \tilde{\mathbf{b}}\ ] \quad (\text{ or } [\ \mathrm{RREF}(X) \mid \tilde{\mathbf{y}}\ ]\ )$$

4^{th}, interpret $[\ A \mid \mathbf{b}\ ]$ (or $[\ X \mid \mathbf{y}\ ]$) to determine interpolating function $f(x)$:

Notice that each column of $\mathrm{RREF}(A)$ (or $\mathrm{RREF}(X)$) has a **pivot**,

which means the solution to the linear system $\mathbf{x}$ (or $\mathbf{c}$) is **unique**,

which means the interpolating function $f(x)$ is **unique**.

$$\therefore \underbrace{\mathbf{x}}_{=\mathbf{c}} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \\ 1/2 \\ 0 \end{bmatrix} \Longrightarrow \boxed{f(x) = 7 - 3x + \frac{1}{2}x^2}$$