CONSTRUCTING $A\mathbf{x} = \mathbf{b}$: CURVE INTERPOLATION [LARSON 1.3]

- **CURVE INTERPOLATION (DEFINITION):**

  Let function $f(x)$ be **continuous**. Then:

  The curve $y = f(x)$ **interpolates** a set of n points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ if:

  $$f(x_1) = y_1, \quad f(x_2) = y_2, \quad \cdots, \quad f(x_n) = y_n$$

  i.e. A curve **interpolates** a set of points if the curve **contains** all the points.

- **POLYNOMIAL INTERPOLATION:**

  Given n points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ s.t. all x-coordinates are **distinct**.

  Then, there exists a **unique** $(n-1)$-degree interpolating polynomial

  $$p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_{n-1} x^{n-1}$$

  where scalars $c_0, c_1, c_2, \ldots, c_{n-1} \in \mathbb{R}$ are to be determined such that

  $$p(x_1) = y_1, \quad p(x_2) = y_2, \quad \ldots, \quad p(x_n) = y_n$$

  For instance, there's a unique quadratic $p(x) = c_0 + c_1 x + c_2 x^2$ that contains the **three** points $(-1, 4), (0, -2), (3, 5)$.

  There's a unique cubic $p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$ that contains **four** points, etc...

- **POLYNOMIAL INTERPOLATION (PROCEDURE):**

  GIVEN: Points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ s.t. all x-coordinates are **distinct**.

  TASK: Find unique interpolating polynomial $p(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_{n-1} x^{n-1}$

  (1) Setup linear system $A\mathbf{x} = \mathbf{b}$ using $p(x_1) = y_1, \ p(x_2) = y_2, \ \ldots, \ p(x_n) = y_n$:

  $$\begin{array}{ccccccccccccc} c_0 & + & c_1 x_1 & + & c_2 x_1^2 & + & c_3 x_1^3 & + & \cdots & + & c_{n-1} x_1^{n-1} & = & y_1 \\ c_0 & + & c_1 x_2 & + & c_2 x_2^2 & + & c_3 x_2^3 & + & \cdots & + & c_{n-1} x_2^{n-1} & = & y_2 \\ \vdots & & \vdots & & \vdots & & \vdots & & \ddots & & & & \vdots \\ c_0 & + & c_1 x_n & + & c_2 x_n^2 & + & c_3 x_n^3 & + & \cdots & + & c_{n-1} x_n^{n-1} & = & y_n \end{array}$$

  This is a $n \times n$ square linear system with unknowns $c_0, c_1, c_2, \ldots, c_{n-1}$.

  (2) Solve linear system using Gauss-Jordan Elimination as usual.

- **DIFFERENTIAL CURVE INTERPOLATION (DEFINITION):**

  Let function $f \in C^{n-1}$. Then the curve $y = f(x)$ **interpolates** a point (x_0, y_0) **in the differential sense** if:

  $$f(x_0) = y_0, \quad f'(x_0) = \alpha_1, \quad f''(x_0) = \alpha_2, \quad f'''(x_0) = \alpha_3, \quad f^{(4)}(x_0) = \alpha_4, \quad \cdots, \quad f^{(n-1)}(x_0) = \alpha_{n-1}$$

  where scalars $\alpha_1, \alpha_2, \ldots, \alpha_{n-1} \in \mathbb{R}$.

  i.e. The curve contains a single point but also must satisfy prescribed derivative values at that point.

- **DIFFERENTIAL CURVE INTERPOLATION (PROCEDURE):**

GIVEN: Point (x_0, y_0) & function $f \in C^{n-1}$ s.t. $f(x) = c_1 f_1(x) + \cdots + c_n f_n(x)$

TASK: Find coefficients $c_1, \ldots, c_n$ s.t. f satisfies the following conditions:

$$f(x_0) = y_0, \quad f'(x_0) = \alpha_1, \quad f''(x_0) = \alpha_2, \quad f'''(x_0) = \alpha_3, \quad f^{(4)}(x_0) = \alpha_4, \quad \cdots, \quad f^{(n-1)}(x_0) = \alpha_{n-1}$$

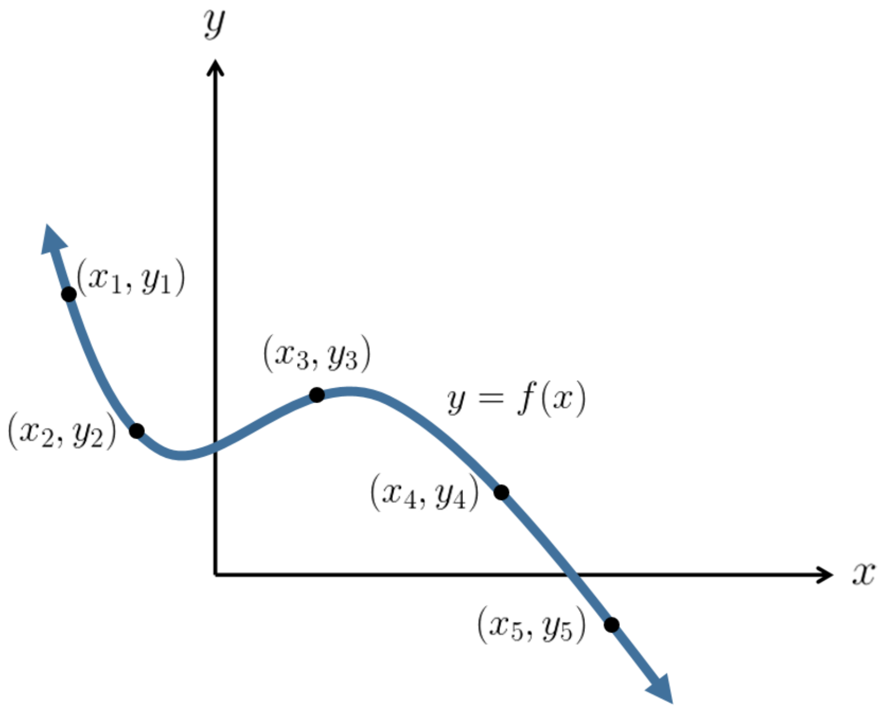
(1) Setup $n \times n$ linear system where each equation satisfies a condition.

(2) Solve linear system using Gauss-Jordan Elimination as usual.

($\star$) For simplicity, functions $f_1(x), f_2(x), \ldots, f_n(x)$ can only be:

  - Polynomials: $1, x, x^2, x^3, \ldots$
  - Exponentials: $e^x, e^{-x}, e^{2x}, e^{-2x}, \ldots$
  - Sines/Cosines: $\sin x, \cos x, \sin 2x, \cos 2x, \ldots$
  - Products of these: $xe^x, x \sin 2x, x^3 \cos x, e^{2x} \sin 3x, \ldots$

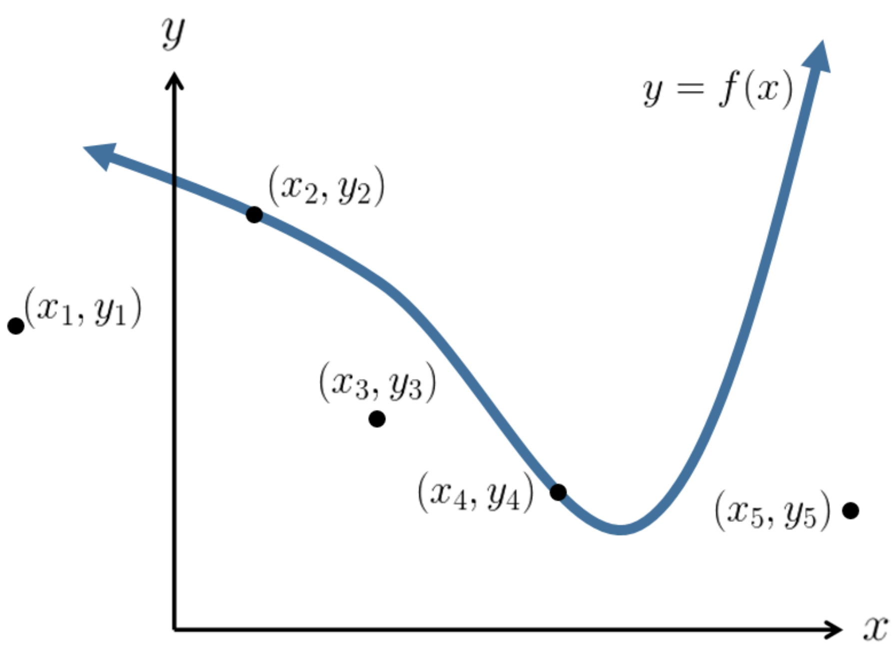
NOTATION: $f \in C^n$ means function f is n-times continuously differentiable.



This is Curve Interpolation

since curve $y = f(x)$ contains points $(x_1, y_1), \ldots, (x_5, y_5)$

i.e. $f(x_1) = y_1,\ f(x_2) = y_2,\ f(x_3) = y_3,\ f(x_4) = y_4,\ f(x_5) = y_5$



This is **NOT** Curve Interpolation

since curve $y = f(x)$ does **not** contain points $(x_1, y_1), (x_3, y_3), (x_5, y_5)$

i.e. $f(x_1) \neq y_1, \quad f(x_3) \neq y_3, \quad f(x_5) \neq y_5$

EX 1.3.1: Find a quadratic polynomial $p(x) = c_0 + c_1 x + c_2 x^2$ such that it contains the points $(-1, 4), (0, -2), (3, 5)$.

In other words, p must satisfy: $p(-1) = 4, \quad p(0) = -2, \quad p(3) = 5.$

EX 1.3.2: Find a function $f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$ such that: $f(2) = 3, f'(2) = -1, f''(2) = 1, f'''(2) = 0.$

EX 1.3.3: Find a function $g(x) = c_0 + c_1 \sin(2x) + c_2 \cos(2x)$ such that: $g(\pi/4) = 4, g'(\pi/4) = 0, g''(\pi/4) = -2.$