# MATRIX ALGEBRA: ADD, TRANSPOSE, MULTIPLY [LARSON 2.1]

- **EQUALITY OF MATRICES:**

Let matrices $A = [a_{ij}], B = [b_{ij}]$. Then $A$ and $B$ are **equal** $\iff$ $A, B$ have the **same shape** and $a_{ij} = b_{ij} \quad \forall i, j$

$$\begin{bmatrix} 1 & 3 & \pi \\ \sqrt{2} & -1 & 7 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & x \\ \sqrt{2} & y & 7 \\ z & 0 & 1 \end{bmatrix} \iff \begin{cases} x & = & \pi \\ y & = & -1 \\ z & = & 0 \end{cases}$$

- **MATRIX ADDITION/SUBTRACTION:**

Let matrices $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$. Then $A + B := [a_{ij} + b_{ij}]_{m\times n} \qquad A - B := [a_{ij} - b_{ij}]_{m\times n}$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 1 & 3 & \pi \\ 7 & -1 & \sqrt{3} \\ 4 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 & (3+\pi) \\ 11 & 4 & (6+\sqrt{3}) \\ 11 & 13 & 10 \end{bmatrix}$$

- **SCALAR MULTIPLICATION:**

Let matrix $A = [a_{ij}]_{m\times n}$ and scalar $\alpha \in \mathbb{R}$. Then $\alpha A := [\alpha a_{ij}]_{m\times n}$

$$(-2)\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} (-2)(1) & (-2)(2) & (-2)(3) \\ (-2)(4) & (-2)(5) & (-2)(6) \end{bmatrix} = \begin{bmatrix} -2 & -4 & -6 \\ -8 & -10 & -12 \end{bmatrix}$$

- **TRANSPOSE:**

Let matrix $A = [a_{ij}]_{m\times n}$. Then $A^T := [a_{ji}]_{n\times m}$

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix},\ B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix},\ \mathbf{v} = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \implies A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix},\ B^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix},\ \mathbf{v}^T = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

- **MATRIX MULTIPLICATION:**

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$. Then $AB := \left[\sum_{k=1}^{n} a_{ik}b_{kj}\right]_{m\times p}$, where $\sum_{k=1}^{n} a_{ik}b_{kj} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}$

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{2\times 2}\underbrace{\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}}_{2\times 2} = \begin{bmatrix} (1)(5)+(2)(7) & (1)(6)+(2)(8) \\ (3)(5)+(4)(7) & (3)(6)+(4)(8) \end{bmatrix} = \underbrace{\begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}}_{2\times 2}$$

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{2\times 2}\underbrace{\begin{bmatrix} 0 & 1 & 2 \\ 4 & 5 & 6 \end{bmatrix}}_{2\times 3} = \begin{bmatrix} (1)(0)+(2)(4) & (1)(1)+(2)(5) & (1)(2)+(2)(6) \\ (3)(0)+(4)(4) & (3)(1)+(4)(5) & (3)(2)+(4)(6) \end{bmatrix} = \underbrace{\begin{bmatrix} 8 & 11 & 14 \\ 16 & 23 & 30 \end{bmatrix}}_{2\times 3}$$

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}}_{3\times 2}\underbrace{\begin{bmatrix} 7 \\ 8 \end{bmatrix}}_{2\times 1} = \underbrace{\begin{bmatrix} 23 \\ 53 \\ 83 \end{bmatrix}}_{3\times 1} \qquad \underbrace{\begin{bmatrix} 7 & 8 & 9 \end{bmatrix}}_{1\times 3}\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}}_{3\times 2} = \underbrace{\begin{bmatrix} 76 & 100 \end{bmatrix}}_{1\times 2}$$

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}}_{1\times 4}\underbrace{\begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}}_{4\times 1} = (1)(5)+(2)(6)+(3)(7)+(4)(8) = \underbrace{70}_{1\times 1}$$

$$\underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}}_{4\times 1}\underbrace{\begin{bmatrix} 5 & 6 & 7 & 8 \end{bmatrix}}_{1\times 4} = \begin{bmatrix} (1)(5) & (1)(6) & (1)(7) & (1)(8) \\ (2)(5) & (2)(6) & (2)(7) & (2)(8) \\ (3)(5) & (3)(6) & (3)(7) & (3)(8) \\ (4)(5) & (4)(6) & (4)(7) & (4)(8) \end{bmatrix} = \underbrace{\begin{bmatrix} 5 & 6 & 7 & 8 \\ 10 & 12 & 14 & 16 \\ 15 & 18 & 21 & 24 \\ 20 & 24 & 28 & 32 \end{bmatrix}}_{4\times 4}$$

**EX 2.1.1:** Let $A = \begin{bmatrix} -3 & 1 \\ 6 & 8 \\ 2 & -4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & -2 & -7 \\ 7 & 1 & -5 \end{bmatrix}$, $C = \begin{bmatrix} 1 & -7 \\ 5 & 0 \\ 8 & -1 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 9 & 3 & 6 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 6 \\ -4 \\ 0 \end{bmatrix}$.

Compute each expression if well-defined, otherwise state "undefined".

$A + B =$

$A + B^T =$

$A - C =$

$A - C^T =$

$\mathbf{u} + \mathbf{v} =$

$\mathbf{u} + \mathbf{v}^T =$

$\mathbf{u}^T - \mathbf{v} =$

$\mathbf{u}^T - \mathbf{v}^T =$

$3A =$

$-\frac{1}{2}B^T =$

$-5\mathbf{u}^T =$

$\pi\mathbf{v}^T =$

$2A + B =$

$2A + B^T =$

$2\mathbf{u}^T + 3\mathbf{v} =$

$\sqrt{7}\mathbf{u} - \pi\mathbf{v}^T =$

**EX 2.1.2:** Let $A = \begin{bmatrix} -2 & 1 \\ 1 & 1 \\ -2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -2 & -2 \\ -3 & 3 & 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 6 \\ -4 \end{bmatrix}$.

Compute each expression if well-defined, otherwise state "undefined".

$AB =$

$BA =$

$A^T B =$

$AB^T =$

$A^T A =$

$AA^T =$

$A\mathbf{v} =$

$\mathbf{v}^T A =$

$\mathbf{v}^T \mathbf{v} =$

$\mathbf{v}\mathbf{v}^T =$

**EX 2.1.3:** Let $A = \begin{bmatrix} -2 & 1 & 3 \\ 1 & 1 & -4 \\ -2 & 2 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -2 & -2 \\ -3 & 3 & 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 6 \\ -4 \\ 0 \end{bmatrix}$.

Compute each expression if well-defined, otherwise state "undefined".

$AB =$

$BA =$

$A^T B =$

$AB^T =$

$B^T A =$

$BA^T =$

$A\mathbf{v} =$

$\mathbf{v}^T A =$