MATRIX ALGEBRA: PROPERTIES, ZERO/IDENTITY/DIAGONAL MATRICES [LARSON 2.2]

- **ZERO MATRIX:** The $m \times n$ matrix O is called the $m \times n$ **zero matrix** if $O = [0]_{m\times n}$

$$O_{2\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, O_{3\times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, O_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, O_{3\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \cdots$$

- **PROPERTIES OF MATRIX ADDITION & SCALAR MULTIPLICATION:**

Let A, B, C be $m \times n$ matrices, O be the $m \times n$ zero matrix, and α, β be scalars. Then:

(A1) $A + B = B + A$ (A2) $A + (B + C) = (A + B) + C$

(A3) $A + O = A$ (A4) $A + (-A) = O$

(A5) $(\alpha\beta)A = \alpha(\beta A)$ (A6) $(1)A = A$

(A7) $\alpha(A + B) = \alpha A + \alpha B$ (A8) $(\alpha + \beta)A = \alpha A + \beta A$

- **KRONECKER DELTA:** $\delta_{ij} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$, where i, j are **postive integers**

- **IDENTITY MATRIX:** The $n \times n$ **square** matrix I is called the $n \times n$ **identity matrix** if

$$I = [\delta_{ij}]_{n\times n} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}, I_{2\times 2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, I_{3\times 3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, I_{4\times 4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \cdots$$

- **PROPERTIES OF MATRIX MULTIPLICATION:**

Let A, B, C be matrices s.t. the given matrix products are well-defined. Let α be a scalar. Then:

(M1) $A(BC) = (AB)C$ (M2) $\alpha(AB) = (\alpha A)B = A(\alpha B)$

(M3) $A(B + C) = AB + AC$ (M4) $(B + C)A = BA + CA$

NOTE: Since $AB \neq BA$ in general, $A(B + C) \neq (B + C)A$ in general.

- **PROPERTIES OF TRANSPOSES:**

Let A, B be matrices s.t. the given matrix sums/products are well-defined.

Moreover, let α be a scalar. Then:

(T1) $(A^T)^T = A$ (T2) $(A + B)^T = A^T + B^T$

(T3) $(\alpha A)^T = \alpha(A^T)$ (T4) $(AB)^T = B^T A^T$

- **POWERS OF A SQUARE MATRIX:**

Let A be a $n \times n$ **square** matrix and I be the $n \times n$ **identity** matrix. Moreover, let k be a **positive integer**. Then:

$$A^0 := I, \quad A^1 := A, \quad A^2 := AA, \quad A^3 := AAA, \quad A^k := \underbrace{AA \cdots A}_{k \text{ factors}}$$

- **PROPERTIES OF POWERS OF SQUARE MATRICES:**

Let A be a $n \times n$ **square** matrix and j, k be **nonnegative integers**. Then:

(P1) $A^j A^k = A^{j+k}$ (P2) $(A^j)^k = A^{jk}$

- **DIAGONAL MATRIX AND ITS POWERS:** D is a $n \times n$ **diagonal matrix** and k is a **nonnegative integer**.

$$D = [d_{ij}\delta_{ij}]_{n\times n} = \begin{bmatrix} d_{11} & 0 & \cdots & 0 \\ 0 & d_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn} \end{bmatrix}, \qquad D^k = [d_{ij}^k \delta_{ij}]_{n\times n} = \begin{bmatrix} d_{11}^k & 0 & \cdots & 0 \\ 0 & d_{22}^k & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_{nn}^k \end{bmatrix}$$

EX 2.2.1: Suppose $A\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

(a) If $\mathbf{v}$ is a 5×1 column vector, what shape must matrix A be?

(b) If A is a 3×4 matrix, what shape must column vector $\mathbf{v}$ be?

(c) Compute $A(-8\mathbf{v})$

(d) Compute $A\left(\frac{1}{4}\mathbf{v}\right)$

EX 2.2.2: Let X be a 2×2 matrix and O be the 2×2 zero matrix.

Solve the following matrix equation for X: $$2X + \begin{bmatrix} 2 & -2 \\ 4 & 8 \end{bmatrix} + 8\begin{bmatrix} 1 & 0 \\ -1/2 & 0 \end{bmatrix} = O$$

EX 2.2.3: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Of the possible matrix products $ABC, ACB, BAC, BCA, CAB, CBA$, which are well-defined and what are their shapes?

EX 2.2.4: Let $A = \begin{bmatrix} 1 & 1 \\ 2 & -3 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}$.

(a) Compute A^3

(b) Compute D^3