EX 2.3.1: Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$.

(a) Find A^{-1}.

Since A is 2×2, don't bother with Gauss-Jordan Elimination. Instead use the easier formula:

$$A^{-1} = \frac{1}{(1)(4)-(-1)(2)} \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 2/3 & 1/6 \\ -1/3 & 1/6 \end{bmatrix}}$$

(b) Use A^{-1} to solve $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

$$A\mathbf{x} = \mathbf{b} \implies \mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{6} \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} (4)(1)+(1)(3) \\ (-2)(1)+(1)(3) \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 7 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} 7/6 \\ 1/6 \end{bmatrix}}$$

REMARK: Notice that $\frac{1}{6}$ was factored from A^{-1} to avoid messy fraction arithmetic when multiplying.

(c) Use A^{-1} to solve $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$.

$$A\mathbf{x} = \mathbf{b} \implies \mathbf{x} = A^{-1}\mathbf{b} = \frac{1}{6} \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} (4)(-2)+(1)(-1) \\ (-2)(-2)+(1)(-1) \end{bmatrix} = \frac{1}{6} \begin{bmatrix} -9 \\ 3 \end{bmatrix} = \boxed{\begin{bmatrix} -3/2 \\ 1/2 \end{bmatrix}}$$

REMARK: Notice that $\frac{1}{6}$ was factored from A^{-1} to avoid messy fraction arithmetic when multiplying.

(d) Use A^{-1} to solve $AX = B$, where $B = \begin{bmatrix} 1 & 5 \\ 0 & 2 \end{bmatrix}$.

$$AX = B \implies X = A^{-1}B = \frac{1}{6} \begin{bmatrix} 4 & 1 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 0 & 2 \end{bmatrix} = \frac{1}{6} \begin{bmatrix} 4 & 22 \\ -2 & -8 \end{bmatrix} = \boxed{\begin{bmatrix} 2/3 & 11/3 \\ -1/3 & -4/3 \end{bmatrix}}$$

EX 2.3.2: Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 4 \\ 3 & 6 & 5 \end{bmatrix}$. Find A^{-1} if it exists.

$$[A|I] = \left[\begin{array}{ccc|ccc} \boxed{1} & 1 & 1 & 1 & 0 & 0 \\ 3 & 5 & 4 & 0 & 1 & 0 \\ 3 & 6 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow[(-3)R_1+R_3\to R_3]{(-3)R_1+R_2\to R_2} \left[\begin{array}{ccc|ccc} \boxed{1} & 1 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & -3 & 1 & 0 \\ 0 & 3 & 2 & -3 & 0 & 1 \end{array}\right] \xrightarrow[2R_3\to R_3]{3R_2\to R_2} \left[\begin{array}{ccc|ccc} \boxed{1} & 1 & 1 & 1 & 0 & 0 \\ 0 & 6 & 3 & -9 & 3 & 0 \\ 0 & 6 & 4 & -6 & 0 & 2 \end{array}\right]$$

$$\xrightarrow{(-1)R_2+R_3\to R_3} \left[\begin{array}{ccc|ccc} \boxed{1} & 1 & 1 & 1 & 0 & 0 \\ 0 & 6 & 3 & -9 & 3 & 0 \\ 0 & 0 & 1 & 3 & -3 & 2 \end{array}\right] \xrightarrow[(-1)R_3+R_1\to R_1]{(-3)R_3+R_2\to R_2} \left[\begin{array}{ccc|ccc} \boxed{1} & 1 & 0 & -2 & 3 & -2 \\ 0 & 6 & 0 & -18 & 12 & -6 \\ 0 & 0 & 1 & 3 & -3 & 2 \end{array}\right]$$

$$\xrightarrow{(\frac{1}{6})R_2\to R_2} \left[\begin{array}{ccc|ccc} \boxed{1} & 1 & 0 & -2 & 3 & -2 \\ 0 & \boxed{1} & 0 & -3 & 2 & -1 \\ 0 & 0 & \boxed{1} & 3 & -3 & 2 \end{array}\right] \xrightarrow{(-1)R_2+R_1\to R_1} \left[\begin{array}{ccc|ccc} \boxed{1} & 0 & 0 & 1 & 1 & -1 \\ 0 & \boxed{1} & 0 & -3 & 2 & -1 \\ 0 & 0 & \boxed{1} & 3 & -3 & 2 \end{array}\right] = [I|A^{-1}]$$

$$\therefore\ A^{-1} = \boxed{\begin{bmatrix} 1 & 1 & -1 \\ -3 & 2 & -1 \\ 3 & -3 & 2 \end{bmatrix}}$$

EX 2.3.3: Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 3 & 6 \\ 2 & 4 & 6 \end{bmatrix}$. Find A^{-1} if it exists.

$$[\ A\ |\ I\ ] = \left[\begin{array}{ccc|ccc} \boxed{1} & 1 & 2 & 1 & 0 & 0 \\ 3 & 3 & 6 & 0 & 1 & 0 \\ 2 & 4 & 6 & 0 & 0 & 1 \end{array}\right] \xrightarrow[(-2)R_1+R_3\to R_3]{(-3)R_1+R_2\to R_2} \left[\begin{array}{ccc|ccc} \boxed{1} & 1 & 2 & 1 & 0 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{-3} & \mathbf{1} & \mathbf{0} \\ 0 & 2 & 2 & -2 & 0 & 1 \end{array}\right]$$

Translating 2^{nd} row so far yields:

$0 = -3$ ←CONTRADICTION!

$0 = 1$ ←CONTRADICTION!

$0 = 0$ ← TAUTOLOGY

Therefore, since there's at least one CONTRADICTION, RREF$(A) \neq I \implies A$ has no inverse $\implies$ **A is singular**

The point is when performing Gauss-Jordan Elimination on augmented matrix $[A|I]$, if at any point of the procedure you have **a row with all zeroes left of the vertical bar and at least one non-zero right of the vertical bar**, you can stop and conclude there's at least one CONTRADICTION, hence RREF$(A) \neq I$ which implies the matrix has **no inverse**.

EX 2.3.4: Let $A^{-1} = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & 2 \\ 1 & 0 & 0 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} -1 & 4 & 4 \\ 0 & -1 & -2 \\ 3 & -2 & 4 \end{bmatrix}$.

(a) Compute $\left(-\frac{1}{3}A\right)^{-1}$.

$$\left(-\frac{1}{3}A\right)^{-1} \overset{I3}{=} (-3)A^{-1} = (-3)\begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & 2 \\ 1 & 0 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} -3 & 0 & -9 \\ 3 & -6 & -6 \\ -3 & 0 & 0 \end{bmatrix}}$$

(b) Compute $(B^2)^{-1}$.

$$(B^2)^{-1} \overset{I2}{=} (B^{-1})^2 = B^{-1}B^{-1} = \begin{bmatrix} -1 & 4 & 4 \\ 0 & -1 & -2 \\ 3 & -2 & 4 \end{bmatrix}\begin{bmatrix} -1 & 4 & 4 \\ 0 & -1 & -2 \\ 3 & -2 & 4 \end{bmatrix} = \boxed{\begin{bmatrix} 13 & -16 & 4 \\ -6 & 5 & -6 \\ 9 & 6 & 32 \end{bmatrix}}$$

(c) Compute $(AB)^{-1}$.

$$(AB)^{-1} \overset{I5}{=} B^{-1}A^{-1} = \begin{bmatrix} -1 & 4 & 4 \\ 0 & -1 & -2 \\ 3 & -2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & 2 \\ 1 & 0 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} -1 & 8 & 5 \\ -1 & -2 & -2 \\ 9 & -4 & 5 \end{bmatrix}}$$