SOLVING SQUARE $A\mathbf{x} = \mathbf{b}$: INVERSE OF A SQUARE MATRIX [LARSON 2.3]

- **INVERSE OF A SQUARE MATRIX:** Let A be a $n \times n$ **square** matrix and I be the $n \times n$ **identity** matrix.

  Then A is **invertible** if there exists a $n \times n$ matrix A^{-1} such that $A^{-1}A = AA^{-1} = I$

  If A does not have an inverse, A is called **singular** (AKA **noninvertible**).

  Non-square matrices do not have inverses (since $AB \neq BA$ if $m \neq n$.)

- **UNIQUENESS OF AN INVERSE:** If $n \times n$ square matrix A is invertible, then its inverse A^{-1} is **unique**.

- **FINDING AN INVERSE MATRIX VIA GAUSS-JORDAN ELIMINATION:**

  GIVEN: **Square** $n \times n$ matrix A.

  TASK: Find A^{-1} if it exists, otherwise conclude A is singular.

  (1) Form **augmented** matrix $[A|I]$, where I is $n \times n$ **identity** matrix.

  (2) Apply **Gauss-Jordan Elimination** to $[A|I]$:

  If RREF$(A) \neq I$, then A is **singular**.

  If RREF$(A) = I$, then $[A|I] \xrightarrow{Gauss-Jordan} [I|A^{-1}]$

  WARNING: There will **never** be a **pivot** to the **right of vertical divider**.

  SANITY CHECK: Check that $A^{-1}A = I$ and $AA^{-1} = I$.

- **INVERSE OF A 2×2 MATRIX:**

  Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ be a 2×2 matrix s.t. $a, b, c, d \in \mathbb{R}$. Then:

  If $ad - bc \neq 0$, then $A^{-1} = \dfrac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

  If $ad - bc = 0$, then A is **singular**.

- **PROPERTIES OF INVERSES:**

  Let A, B be $n \times n$ **invertible** matrices, k be **positive integer**, and $\alpha \neq 0$.

  Then A^{-1}, A^k, αA, A^T, AB are all invertible and the following are true:

(I1)	$(A^{-1})^{-1} = A$	Inverse of an Inverse
(I2)	$(A^k)^{-1} = (A^{-1})^k$	Inverse of a Power
(I3)	$(\alpha A)^{-1} = \frac{1}{\alpha} A^{-1}$	Inverse of a Scalar Mult.
(I4)	$(A^T)^{-1} = (A^{-1})^T$	Inverse of a Transpose
(I5)	$(AB)^{-1} = B^{-1}A^{-1}$	Inverse of a Product

- **INVERSE OF AN EXTENDED PRODUCT:**

  Let A, B, C be $n \times n$ **invertible** matrices. Then: $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

  Let $A_1, A_2, \ldots, A_{k-1}, A_k$ be $n \times n$ **invertible** matrices. Then: $(A_1A_2 \cdots A_{k-1}A_k)^{-1} = A_k^{-1}A_{k-1}^{-1} \cdots A_2^{-1}A_1^{-1}$

- **CANCELLATION PROPERTIES OF MATRIX PRODUCTS:**

  Let C be an **invertible** matrix and A, B have compatible shapes. Then:

(C1)	If $AC = BC$, then $A = B$	Right-cancellation
(C2)	If $CA = CB$, then $A = B$	Left-cancellation

- **SOLVING SQUARE LINEAR SYSTEM $A\mathbf{x} = \mathbf{b}$ VIA A^{-1}:** $\mathbf{x} = A^{-1}\mathbf{b}$

- **SOLVING SQUARE LINEAR SYSTEM $AX = B$ VIA A^{-1}:** $X = A^{-1}B$

- **SOLVING SQUARE LINEAR SYSTEM $XA = B$ VIA A^{-1}:** $X = BA^{-1}$

EX 2.3.1: Let $A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}$.

(a) Find A^{-1}.

(b) Use A^{-1} to solve $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

(c) Use A^{-1} to solve $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$.

(d) Use A^{-1} to solve $AX = B$, where $B = \begin{bmatrix} 1 & 5 \\ 0 & 2 \end{bmatrix}$.

EX 2.3.2: Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 5 & 4 \\ 3 & 6 & 5 \end{bmatrix}$. Find A^{-1} if it exists.

EX 2.3.3: Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 3 & 6 \\ 2 & 4 & 6 \end{bmatrix}$. Find A^{-1} if it exists.

EX 2.3.4: Let $A^{-1} = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 2 & 2 \\ 1 & 0 & 0 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} -1 & 4 & 4 \\ 0 & -1 & -2 \\ 3 & -2 & 4 \end{bmatrix}$.

(a) Compute $\left(-\frac{1}{3}A\right)^{-1}$.

(b) Compute $\left(B^2\right)^{-1}$.

(c) Compute $(AB)^{-1}$.