EX 2.4.1: Let $A = \begin{bmatrix} -2 & -1 & 0 \\ 8 & 9 & 1 \\ -6 & 7 & 8 \end{bmatrix}$.

(a) Find the LU-Factorization of A.

Apply COMBINE elem. row operations to zero out entries below the main diagonal, resulting in U:

$$\xrightarrow{4R_1+R_2\to R_2} \begin{bmatrix} -2 & -1 & 0 \\ 0 & 5 & 1 \\ -6 & 7 & 8 \end{bmatrix} = E_1A \implies E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \implies E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{(-3)R_1+R_3\to R_3} \begin{bmatrix} -2 & -1 & 0 \\ 0 & 5 & 1 \\ 0 & 10 & 8 \end{bmatrix} = E_2E_1A \implies E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \implies E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{(-2)R_2+R_3\to R_3} \begin{bmatrix} -2 & -1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 6 \end{bmatrix} = E_3E_2E_1A \implies E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \implies E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$\therefore \quad U = \begin{bmatrix} -2 & -1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 6 \end{bmatrix} \quad \text{and} \quad E_3E_2E_1A = U \implies A = \underbrace{E_1^{-1}E_2^{-1}E_3^{-1}}_{L}U$$

$$\therefore \quad L = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

$$\therefore \quad \boxed{\begin{bmatrix} -2 & -1 & 0 \\ 8 & 9 & 1 \\ -6 & 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} -2 & -1 & 0 \\ 0 & 5 & 1 \\ 0 & 0 & 6 \end{bmatrix}}$$

(b) Use $A = LU$ to solve linear system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} -8 \\ 23 \\ -36 \end{bmatrix}$.

$$L\mathbf{y} = \mathbf{b} \implies \begin{cases} y_1 & & & = -8 \\ -4y_1 & + \; y_2 & & = 23 \\ 3y_1 & + \; 2y_2 & + \; y_3 & = -36 \end{cases} \xrightarrow{\text{Forward-Solve}} \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -8 \\ -9 \\ 6 \end{bmatrix}$$

$$U\mathbf{x} = \mathbf{y} \implies \begin{cases} -2x_1 & - \; x_2 & & = -8 \\ & 5x_2 & + \; x_3 & = -9 \\ & & 6x_3 & = 6 \end{cases} \xrightarrow{\text{Back-Solve}} \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \boxed{\begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}}$$

(c) Use $A = LU$ to solve linear system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} 6 \\ -25 \\ 10 \end{bmatrix}$.

$$L\mathbf{y} = \mathbf{b} \implies \begin{cases} y_1 & & & = 6 \\ -4y_1 & + \; y_2 & & = -25 \\ 3y_1 & + \; 2y_2 & + \; y_3 & = 10 \end{cases} \xrightarrow{\text{Forward-Solve}} \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ -6 \end{bmatrix}$$

$$U\mathbf{x} = \mathbf{y} \implies \begin{cases} -2x_1 & - \; x_2 & & = 6 \\ & 5x_2 & + \; x_3 & = -1 \\ & & 6x_3 & = -6 \end{cases} \xrightarrow{\text{Back-Solve}} \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \boxed{\begin{bmatrix} -3 \\ 0 \\ -1 \end{bmatrix}}$$

EX 2.4.2: Let $A = \begin{bmatrix} 3 & 1 & 0 & -1 & 1 \\ -12 & -2 & 4 & 5 & 1 \\ 9 & 9 & 9 & 0 & 24 \\ 0 & -4 & -11 & 2 & -5 \end{bmatrix}$. Find the LU-Factorization of A.

Apply COMBINE elem. row operations to zero out entries below the main diagonal, resulting in U:

$$\xrightarrow{4R_1+R_2\to R_2} \begin{bmatrix} 3 & 1 & 0 & -1 & 1 \\ 0 & 2 & 4 & 1 & 5 \\ 9 & 9 & 9 & 0 & 24 \\ 0 & -4 & -11 & 2 & -5 \end{bmatrix} \implies E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \implies E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{(-3)R_1+R_3\to R_3} \begin{bmatrix} 3 & 1 & 0 & -1 & 1 \\ 0 & 2 & 4 & 1 & 5 \\ 0 & 6 & 9 & 3 & 21 \\ 0 & -4 & -11 & 2 & -5 \end{bmatrix} \implies E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \implies E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{(-3)R_2+R_3\to R_3} \begin{bmatrix} 3 & 1 & 0 & -1 & 1 \\ 0 & 2 & 4 & 1 & 5 \\ 0 & 0 & -3 & 0 & 6 \\ 0 & -4 & -11 & 2 & -5 \end{bmatrix} \implies E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \implies E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{2R_2+R_4\to R_4} \begin{bmatrix} 3 & 1 & 0 & -1 & 1 \\ 0 & 2 & 4 & 1 & 5 \\ 0 & 0 & -3 & 0 & 6 \\ 0 & 0 & -3 & 4 & 5 \end{bmatrix} \implies E_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix} \implies E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{(-1)R_3+R_4\to R_4} \begin{bmatrix} 3 & 1 & 0 & -1 & 1 \\ 0 & 2 & 4 & 1 & 5 \\ 0 & 0 & -3 & 0 & 6 \\ 0 & 0 & 0 & 4 & -1 \end{bmatrix} \implies E_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \implies E_5^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\therefore \quad U = \begin{bmatrix} 3 & 1 & 0 & -1 & 1 \\ 0 & 2 & 4 & 1 & 5 \\ 0 & 0 & -3 & 0 & 6 \\ 0 & 0 & 0 & 4 & -1 \end{bmatrix} \quad \text{and} \quad E_5E_4E_3E_2E_1A = U \implies A = \underbrace{E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}}_{L}U$$

$$\therefore \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 \\ 0 & -2 & 1 & 1 \end{bmatrix}$$

$$\therefore \quad \boxed{\begin{bmatrix} 3 & 1 & 0 & -1 & 1 \\ -12 & -2 & 4 & 5 & 1 \\ 9 & 9 & 9 & 0 & 24 \\ 0 & -4 & -11 & 2 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -4 & 1 & 0 & 0 \\ 3 & 3 & 1 & 0 \\ 0 & -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 & 0 & -1 & 1 \\ 0 & 2 & 4 & 1 & 5 \\ 0 & 0 & -3 & 0 & 6 \\ 0 & 0 & 0 & 4 & -1 \end{bmatrix}}$$

EX 2.4.3: Let $A = \begin{bmatrix} 6 & -2 & 1 \\ 30 & -6 & 8 \\ -12 & -12 & -15 \\ 18 & -2 & 6 \end{bmatrix}$. Find the LU-Factorization of A.

Apply COMBINE elem. row operations to zero out entries below the main diagonal, resulting in U:

$$\xrightarrow{(-5)R_1+R_2\to R_2} \begin{bmatrix} 6 & -2 & 1 \\ 0 & 4 & 3 \\ -12 & -12 & -15 \\ 18 & -2 & 6 \end{bmatrix} \implies E_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -5 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \implies E_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{2R_1+R_3\to R_3} \begin{bmatrix} 6 & -2 & 1 \\ 0 & 4 & 3 \\ 0 & -16 & -13 \\ 18 & -2 & 6 \end{bmatrix} \implies E_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \implies E_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{(-3)R_1+R_4\to R_4} \begin{bmatrix} 6 & -2 & 1 \\ 0 & 4 & 3 \\ 0 & -16 & -13 \\ 0 & 4 & 3 \end{bmatrix} \implies E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{bmatrix} \implies E_3^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{4R_2+R_3\to R_3} \begin{bmatrix} 6 & -2 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & -1 \\ 0 & 4 & 3 \end{bmatrix} \implies E_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 4 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \implies E_4^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{(-1)R_2+R_4\to R_4} \begin{bmatrix} 6 & -2 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \implies E_5 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \implies E_5^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad U = \begin{bmatrix} 6 & -2 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } E_5E_4E_3E_2E_1A = U \implies A = \underbrace{E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}}_{L}U$$

$$\therefore \quad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ -2 & -4 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad \boxed{\begin{bmatrix} 6 & -2 & 1 \\ 30 & -6 & 8 \\ -12 & -12 & -15 \\ 18 & -2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 \\ -2 & -4 & 1 & 0 \\ 3 & 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 & -2 & 1 \\ 0 & 4 & 3 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}}$$