SOLVING $A\mathbf{x} = \mathbf{b}$ WITH DIFFERENT b's: LU-FACTORIZATION [LARSON 2.4]

- **COMBINE ELEMENTARY ROW OPERATIONS:**

  (COMBINE) $[\alpha R_i + R_j \to R_j]$ Add scalar multiple α of row i to row j

- **ELEMENTARY MATRICES:**

  A square matrix E is an **elementary matrix** if it can be obtained from identity matrix I by a single elem row op.

- **ELEMENTARY MATRICES REPRESENTING COMBINE OPERATIONS:**

  $$[5R_1 + R_2 \to R_2] : \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad [(-5)R_2 + R_3 \to R_3] : \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}$$

- **INVERSE OF AN ELEMENTARY MATRIX:**

  If E is an **elementary matrix**, then E^{-1} exists and is an elementary matrix.

  The inverse "undoes" the elementary row operation.

  $$[5R_1 + R_2 \to R_2] : \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{has inverse} \quad [(-5)R_1 + R_2 \to R_2] : \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- **TRIANGULAR MATRICES:**

  $$\text{Lower Triangular: } \begin{bmatrix} 1 & \mathbf{0} \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 1 & \mathbf{0} & \mathbf{0} \\ 4 & 5 & \mathbf{0} \end{bmatrix}, \begin{bmatrix} 1 & \mathbf{0} \\ 3 & 4 \\ 5 & 0 \end{bmatrix}, \begin{bmatrix} 1 & \mathbf{0} & \mathbf{0} \\ 4 & 5 & \mathbf{0} \\ 0 & 8 & 0 \end{bmatrix}, \cdots$$

  $$\text{Upper Triangular: } \begin{bmatrix} 1 & 2 \\ \mathbf{0} & 4 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 3 \\ \mathbf{0} & 5 & 6 \end{bmatrix}, \begin{bmatrix} 1 & 2 \\ \mathbf{0} & 4 \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \begin{bmatrix} 1 & 0 & 3 \\ \mathbf{0} & 0 & 6 \\ \mathbf{0} & \mathbf{0} & 9 \end{bmatrix}, \cdots$$

- **UNIT TRIANGULAR MATRICES:**

  $$\text{Unit Lower Triangular: } \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ 3 & \mathbf{1} \end{bmatrix}, \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 4 & \mathbf{1} & \mathbf{0} \end{bmatrix}, \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ 3 & \mathbf{1} \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ 1 & \mathbf{1} & \mathbf{0} \\ 0 & 8 & \mathbf{1} \end{bmatrix}, \cdots$$

- **LU-FACTORIZATION OF A MATRIX (PROCEDURE):**

  GIVEN: $m \times n$ matrix A where **no row swaps are necessary**.

  TASK: Form $A = LU$, (L is square unit lower triangular & U is upper triangular)

  (1) COMBINE to zero-out an entry below main diagonal: $[\alpha R_i + R_j \to R_j]$
  (2) Form $m \times m$ elementary matrix corresponding to COMBINE operation: E
  (3) Find the inverse of the elementary matrix: E^{-1}
  (4) Repeat steps (1)-(3) for all such entries, top-to-bottom, left-to-right
  (5) Resulting matrix is upper triangular: $U = E_k E_{k-1} \cdots E_3 E_2 E_1 A$
  (6) Determine L: $E_k E_{k-1} \cdots E_2 E_1 A = U \implies A = \underbrace{E_1^{-1} E_2^{-1} \cdots E_{k-1}^{-1} E_k^{-1}}_{L} U$

- **SOLVING $A\mathbf{x} = \mathbf{b}$ VIA LU-FACTORIZATION:**

  GIVEN: $m \times n$ linear system $A\mathbf{x} = \mathbf{b}$ where **no row swaps are necessary**.

  TASK: Solve linear system via $A = LU$

  (1) Perform LU-Factorization of A (see previous slides)
  (2) Notice $A\mathbf{x} = \mathbf{b} \implies (LU)\mathbf{x} = \mathbf{b} \implies L(U\mathbf{x}) = \mathbf{b} \implies$ Let $\mathbf{y} = U\mathbf{x}$
  (3) Solve **square triangular system** $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$ via forward-solve
  (4) Solve **triangular system** $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$

  – If U is **square**, solve $U\mathbf{x} = \mathbf{y}$ via back-solve
  – If U is non-square, solve $U\mathbf{x} = \mathbf{y}$ via Gauss-Jordan Elimination

EX 2.4.1: Let $A = \begin{bmatrix} -2 & -1 & 0 \\ 8 & 9 & 1 \\ -6 & 7 & 8 \end{bmatrix}$.

(a) Find the LU-Factorization of A.

(b) Use $A = LU$ to solve linear system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} -8 \\ 23 \\ -36 \end{bmatrix}$.

(c) Use $A = LU$ to solve linear system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = \begin{bmatrix} 6 \\ -25 \\ 10 \end{bmatrix}$.

EX 2.4.2: Let $A = \begin{bmatrix} 3 & 1 & 0 & -1 & 1 \\ -12 & -2 & 4 & 5 & 1 \\ 9 & 9 & 9 & 0 & 24 \\ 0 & -4 & -11 & 2 & -5 \end{bmatrix}$. Find the LU-Factorization of A.

EX 2.4.3: Let $A = \begin{bmatrix} 6 & -2 & 1 \\ 30 & -6 & 8 \\ -12 & -12 & -15 \\ 18 & -2 & 6 \end{bmatrix}$. Find the LU-Factorization of A.