EX 3.1.2: Find the determinant of $A = \begin{bmatrix} (1-\lambda) & 3 \\ 2 & (4-\lambda) \end{bmatrix}$, where $\lambda \in \mathbb{R}$.

Since A is 2×2, just use the simple formula to find its determinant rather than a cofactor expansion:

$$\det(A) = (1-\lambda)(4-\lambda) - (3)(2) = (4 - 5\lambda + \lambda^2) - 6 = \boxed{\lambda^2 - 5\lambda - 2}$$

EX 3.1.4: Find the determinant of $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -9 & 3 & 0 & 0 & 0 \\ 8 & -8 & -2 & 0 & 0 \\ 7 & 7 & 7 & 7 & 0 \\ 1 & -7 & 5 & -9 & -1 \end{bmatrix}$.

Observe that A is **triangular**, hence its determinant is simply the **product of the main diagonal entries**:

$$\det(A) = a_{11}a_{22}a_{33}a_{44}a_{55} = (1)(3)(-2)(7)(-1) = \boxed{42}$$

EX 3.1.5: Using a cofactor expansion, find the determinant of $A = \begin{bmatrix} 0 & 3 & -2 & 0 & -1 \\ 0 & -4 & 5 & 0 & 0 \\ 0 & 1 & 0 & -3 & 5 \\ -2 & 0 & 0 & 0 & 3 \\ 0 & 1 & -1 & 0 & -1 \end{bmatrix}$.

Observe that A is a **sparse matrix** (i.e. A has lots of zeros.)

Since A is **sparse**, cofactor-expand along the row or column **with the most zeros**:

$$|A| = (0)C_{11} + (0)C_{21} + (0)C_{31} + (-2)C_{41} + (0)C_{51} \quad \text{(Cofactor Expansion along col 1 of } A\text{)}$$

$$= (-2)(-1)^{4+1} \begin{vmatrix} 3 & -2 & 0 & -1 \\ -4 & 5 & 0 & 0 \\ 1 & 0 & -3 & 5 \\ 1 & -1 & 0 & -1 \end{vmatrix} \quad \text{(Definition of Cofactor } C_{41} \text{ of matrix } A\text{)}$$

$$= (2)\left[(0)C^{[1]}_{13} + (0)C^{[1]}_{23} + (-3)C^{[1]}_{33} + (0)C^{[1]}_{43}\right] \quad \text{(Cofactor Expansion along col 3 of } C_{41}\text{)}$$

$$= (2)(-3)(-1)^{3+3} \begin{vmatrix} 3 & -2 & -1 \\ -4 & 5 & 0 \\ 1 & -1 & -1 \end{vmatrix} \quad \text{(Definition of Cofactor } C^{[1]}_{33} \text{ of matrix } C_{41}\text{)}$$

$$= (-6)\left[(-4)C^{[2]}_{21} + (5)C^{[2]}_{22} + (0)C^{[2]}_{23}\right] \quad \text{(Cofactor Expansion along row 2 of } C^{[1]}_{33}\text{)}$$

$$= (-6)\left[(-4)(-1)^{2+1}\begin{vmatrix} -2 & -1 \\ -1 & -1 \end{vmatrix} + (5)(-1)^{2+2}\begin{vmatrix} 3 & -1 \\ 1 & -1 \end{vmatrix}\right] \quad \text{(Defn's of Cofactors } C^{[2]}_{21} \text{ \& } C^{[2]}_{22} \text{ of matrix } C^{[1]}_{33}\text{)}$$

$$= (-6)\Big[(4)[(-2)(-1) - (-1)(-1)] + (5)[(3)(-1) - (-1)(1)]\Big] \quad \text{(Determinant Formula for } 2 \times 2 \text{ matrices)}$$

$$= (-6)\Big[(4)(1) + (5)(-2)\Big]$$

$$= \boxed{36}$$

REMARK: The superscripts in square brackets of cofactors (e.g. $C^{[2]}_{22}$) are there to prevent abuse of notation.