DETERMINANTS: INTRO & COFACTOR EXPANSIONS [LARSON 3.1]

- **DETERMINANT ("FIRST PRINCIPLES" DEFINTION):**

  Let linear system $A\mathbf{x} = \mathbf{b}$ be **square** & have a **unique** solution.

  Then the denominator of the solution is called the **determinant** of matrix A, denoted $|A|$ or $\det(A)$

  The denominator of a non-square matrix is undefined.

- **DETERMINANT OF A 2×2 SQUARE MATRIX:** Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a, b, c, d \in \mathbb{R}$. Then:

$$\det(A) \equiv \begin{vmatrix} a & b \\ c & d \end{vmatrix} := ad - bc$$

- **MINORS & COFACTORS OF A SQUARE MATRIX:** Let A be a $n \times n$ **square matrix**. Then:

  The (i,j)-**minor** M_{ij} of A is the determinant of matrix obtained by **removing** the i^{th} row & j^{th} column of A.

  The (i,j)-**cofactor** C_{ij} of A is $C_{ij} := (-1)^{i+j} M_{ij}$

e.g. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. Then:

$$C_{11} = (-1)^{1+1} M_{11} = (1)\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} = -3 \qquad C_{12} = (-1)^{1+2} M_{12} = (-1)\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} = 6 \qquad C_{13} = (-1)^{1+3} M_{13} = (1)\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} = -3$$

$$C_{21} = (-1)^{2+1} M_{21} = (-1)\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} = 6 \qquad C_{22} = (-1)^{2+2} M_{22} = (1)\begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} = -12 \qquad C_{23} = (-1)^{2+3} M_{23} = (-1)\begin{vmatrix} 1 & 2 \\ 7 & 8 \end{vmatrix} = 6$$

$$C_{31} = (-1)^{3+1} M_{31} = (1)\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} = -3 \qquad C_{32} = (-1)^{3+2} M_{32} = (-1)\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} = 6 \qquad C_{33} = (-1)^{3+3} M_{33} = (1)\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = -3$$

- **DETERMINANT VIA COFACTOR EXPANSION:** Let A be a $n \times n$ **square matrix**. Then:

$$\det(A) = \sum_{k=1}^{n} a_{ik} C_{ik} = a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots + a_{in} C_{in} \qquad (i^{th} \text{ row cofactor expansion})$$

$$\det(A) = \sum_{k=1}^{n} a_{kj} C_{kj} = a_{1j} C_{1j} + a_{2j} C_{2j} + \cdots + a_{nj} C_{nj} \qquad (j^{th} \text{ column cofactor expansion})$$

e.g. Let $A = \begin{bmatrix} 0 & 2 & 1 \\ 4 & 3 & 3 \\ 1 & 1 & 2 \end{bmatrix}$. Then:

$$\begin{aligned}
|A| &= (0)C_{11} + (2)C_{12} + (1)C_{13} && \text{(Cofactor Expansion along row 1 of } A) \\
&= (0)C_{11} + (4)C_{21} + (1)C_{31} && \text{(Cofactor Expansion along col 1 of } A) \\
&= (4)C_{21} + (3)C_{22} + (3)C_{23} && \text{(Cofactor Expansion along row 2 of } A) \\
&= (2)C_{12} + (3)C_{22} + (1)C_{32} && \text{(Cofactor Expansion along col 2 of } A) \\
&= (1)C_{31} + (1)C_{32} + (2)C_{33} && \text{(Cofactor Expansion along col 3 of } A) \\
&= (1)C_{13} + (3)C_{23} + (2)C_{33} && \text{(Cofactor Expansion along col 3 of } A)
\end{aligned}$$

- **SPARSE MATRICES:** A **sparse matrix** has at least several zeros.
  - Elementary, triangular and diagonal matrices are sparse matrices.
  - Cofactor Expansions are efficient for sparse matrices. It's best to expand along row/column with the **most zeros**

- **DENSE MATRICES:** A **dense matrix** has at most a couple zeros.

- **DETERMINANT OF A TRIANGULAR MATRIX:** Let A be a $n \times n$ **triangular** matrix. Then:

  $\det(A) = a_{11} a_{22} a_{33} \cdots a_{nn}$ (i.e. determinant of a triangular matrix is the product of the diagonal entries)

EX 3.1.1: Find the determinant of $A = \begin{bmatrix} -1 & -4 \\ 20 & 10 \end{bmatrix}$.

EX 3.1.2: Find the determinant of $A = \begin{bmatrix} (1-\lambda) & 3 \\ 2 & (4-\lambda) \end{bmatrix}$, where $\lambda \in \mathbb{R}$.

EX 3.1.3: Using a cofactor expansion, find the determinant of $A = \begin{bmatrix} 5 & 0 & 2 \\ -3 & -1 & 4 \\ -4 & 1 & 6 \end{bmatrix}$.

EX 3.1.4: Find the determinant of $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -9 & 3 & 0 & 0 & 0 \\ 8 & -8 & -2 & 0 & 0 \\ 7 & 7 & 7 & 7 & 0 \\ 1 & -7 & 5 & -9 & -1 \end{bmatrix}$.

EX 3.1.5: Using a cofactor expansion, find the determinant of $A = \begin{bmatrix} 0 & 3 & -2 & 0 & -1 \\ 0 & -4 & 5 & 0 & 0 \\ 0 & 1 & 0 & -3 & 5 \\ -2 & 0 & 0 & 0 & 3 \\ 0 & 1 & -1 & 0 & -1 \end{bmatrix}$.