DETERMINANTS: ELEM. ROW/COL OPERATIONS [LARSON 3.2]

- **ELEMENTARY ROW OPERATIONS & DETERMINANTS:** Let A, B be $n \times n$ square matrices. Then:

$$\begin{array}{llll} \text{(ROW SWAP)} & \text{If} & A \xrightarrow{R_i \leftrightarrow R_j} & B, \text{ then } \det(B) = -\det(A) \\ \text{(ROW SCALE)} & \text{If} & A \xrightarrow{\alpha R_j \to R_j} & B, \text{ then } \det(B) = \alpha\det(A) \\ \text{(ROW COMBINE)} & \text{If} & A \xrightarrow{\alpha R_i + R_j \to R_j} & B, \text{ then } \det(B) = \det(A) \end{array}$$

i.e. Performing a **row swap** causes the determinant to **change sign**.

i.e. Performing a **row scale by** α causes the determinant to **multiplied by** α.

i.e. Performing a **row combine** causes the determinant to **remain the same**.

- **ELEMENTARY COLUMN OPERATIONS & DETERMINANTS:** Let A, B be $n \times n$ square matrices. Then:

$$\begin{array}{llll} \text{(COLUMN SWAP)} & \text{If} & A \xrightarrow{C_i \leftrightarrow C_j} & B, \text{ then } \det(B) = -\det(A) \\ \text{(COLUMN SCALE)} & \text{If} & A \xrightarrow{\alpha C_j \to C_j} & B, \text{ then } \det(B) = \alpha\det(A) \\ \text{(COLUMN COMBINE)} & \text{If} & A \xrightarrow{\alpha C_i + C_j \to C_j} & B, \text{ then } \det(B) = \det(A) \end{array}$$

- **ROW-EQUIVALENT & COLUMN-EQUIVALENT MATRICES:** Let A, B be $n \times n$ matrices. Then:

A and B are **row-equivalent** if B can be obtained from A by elementary row operations.

A and B are **column-equivalent** if B can be obtained from A by elementary column operations.

Moreover, if A and B are row-equivalent or column-equivalent, then $\det(B) = \beta\det(A)$ where $\beta \neq 0$.

- **MATRICES WITH A ZERO DETERMINANT:** Let A be a $n \times n$ square matrix. Then:

(Z1) If an entire row of A consists of all zeros, then $\det(A) = 0$

(Z2) If an entire column of A consists of all zeros, then $\det(A) = 0$

(Z3) If two rows of A are equal, then $\det(A) = 0$

(Z4) If two columns of A are equal, then $\det(A) = 0$

(Z5) If one row of A is a multiple of another row of A, then $\det(A) = 0$

(Z6) If one column of A is a multiple of another column of A, then $\det(A) = 0$

- **FINDING DETERMINANTS VIA ELEMENTARY ROW/COLUMN OPERATIONS:**

GIVEN: $n \times n$ square **dense matrix** A

TASK: Find the determinant of A

(1) Perform elem. row or column op's until one of the following is attained:

  * A matrix with a either a row or column consisting of all zeros (whose determinant is **zero**)
  * A triangular matrix (whose determinant is simply the product of its main diagonal entries)

($\star$) Keep track of each scalar factor resulting from a **SWAP** or **SCALE** operation.

($\star$) For consistency, it's best to use only **row** operations – column operations are never used again in this course.

- **COFACTOR EXPANSIONS VERSUS ELEMENTARY ROW/COLUMN OPERATIONS:**

For large $n \times n$ square **dense matrices** ($n \geq 4$), elem row operations require far less work than a cofactor expansion:

	COFACTOR EXPANSION		**ELEM. ROW OPERATIONS**	
n	ADDITIONS	MULTIPLICATIONS	ADDITIONS	MULTIPLICATIONS
2	1	2	1	3
3	5	9	5	10
4	23	40	14	23
5	119	205	30	44
⋮	⋮	⋮	⋮	⋮
10	**3,628,799**	**6,235,300**	285	339

Even for computers, finding 10×10 determinants are significantly faster using elementary row operations!

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EX 3.2.1: Find the determinants of $A = \begin{bmatrix} 3 & 0 & 2 & 3 & 2 \\ 1 & 0 & 1 & 1 & 1 \\ 8 & 0 & 7 & 6 & 9 \\ 6 & 0 & 8 & 7 & 4 \\ 5 & 0 & 4 & 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 1 & 2 & 3 & 2 \\ 1 & 1 & 1 & 1 & 1 \\ 8 & 7 & 7 & 6 & 9 \\ 6 & 2 & 4 & 6 & 4 \\ 5 & 5 & 4 & 4 & 3 \end{bmatrix}$.

EX 3.2.2: Using elementary row/column operations as appropriate, find the determinant of $A = \begin{bmatrix} -12 & 85 & 60 & 18 \\ -1 & 7 & 5 & 3 \\ 3 & -24 & -21 & 12 \\ 4 & -27 & -17 & -13 \end{bmatrix}$.

EX 3.2.3: Using elementary row/column operations as appropriate, find the determinant of $A = \begin{bmatrix} 8 & 2 & -3 & 1 \\ 32 & -8 & -21 & 12 \\ 4 & -5 & -4 & 3 \\ 160 & -40 & -105 & 60 \end{bmatrix}$.

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