DETERMINANTS: PRODUCTS, INVERSES, TRANSPOSES [LARSON 3.3]

- **DETERMINANTS OF PRODUCTS & TRANSPOSES:** Let A, B be $n \times n$ square matrices and $\alpha \neq 0$. Then:

(D1) $|AB| = |A||B|$ (Determinant of a Matrix Product)

(D2) $|\alpha A| = \alpha^n |A|$ (Determinant of a Scalar Product)

(D3) $|A^T| = |A|$ (Determinant of a Transpose)

- **DETERMINANTS OF EXTENDED MATRIX PRODUCTS:** Let $A_1, A_2, \ldots, A_k$ be $n \times n$ matrices. Then:

(D4) $|A_1 A_2 \cdots A_k| = |A_1||A_2| \cdots |A_k|$ (Determinant of an Extended Product)

- **DETERMINANTS OF POWERS:** Let A be $n \times n$ square matrix and $k \geq 2$ be a **positive integer**. Then:

(D5) $|A^k| = |A|^k$ (Determinant of a Power)

- **DETERMINANT "DETERMINES" INVERTIBILITY OF A MATRIX:**

$$\text{A square matrix } A \text{ is invertible} \iff |A| \neq 0$$

- **DETERMINANTS OF INVERSES:** Let A be a $n \times n$ **invertible** square matrix. Then:

(D6) $|A^{-1}| = \dfrac{1}{|A|}$ (Determinant of an Inverse)

- **DETERMINANTS OF SUMS OR DIFFERENCES (WARNING):** In general, $|A \pm B| \neq |A| \pm |B|$:

$$\text{Consider } A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & -5 \\ 4 & 4 \end{bmatrix} \implies |A| = -2 \text{ and } |B| = 32$$

$$\text{Then } A + B = \begin{bmatrix} 4 & -3 \\ 7 & 8 \end{bmatrix} \text{ and } A - B = \begin{bmatrix} -2 & 7 \\ -1 & 0 \end{bmatrix}$$

$$\implies |A + B| = 53 \neq -2 + 32 = |A| + |B|$$

$$\implies |A - B| = 7 \neq -2 - 32 = |A| - |B|$$

- **EQUIVALENT CONDITIONS FOR AN INVERTIBLE SQUARE MATRIX:**

Let A be a $n \times n$ square matrix. Then the following are equivalent:

· A is invertible

· $A\mathbf{x} = \mathbf{b}$ has a **unique** soln for every RHS column vector $\mathbf{b}$

· $A\mathbf{x} = \vec{\mathbf{0}}$ has only the **trivial** soln $\mathbf{x} = \vec{\mathbf{0}}$ (i.e. $x_1 = 0,\ x_2 = 0,\ \ldots,\ x_n = 0$)

· A is row-equivalent to the identity matrix I

· A can be written as a product of **elementary** matrices

· $|A| \neq 0$

NOTATION: $\vec{\mathbf{0}}$ denotes the **column vector** with all entries being **zero**.

EX 3.3.1: Let $A = \begin{bmatrix} -1 & 28 & -9 & 0 & 54 \\ 0 & 2 & 78 & 99 & 61 \\ 0 & 0 & 1 & -8 & 17 \\ 0 & 0 & 0 & -3 & 77 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 43 & -7 & 16 & -6 \\ 0 & 1 & 88 & 99 & 47 \\ 0 & 0 & -1 & 91 & 81 \\ 0 & 0 & 0 & 2 & 55 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$.

(a) Compute $|A|$ and $|B|$.

(b) Compute $|A^T|$ and $|B^T|$.

(c) Compute $|2A|$ and $\left|\frac{1}{3}B\right|$.

(d) Compute $|AB|$ and $|BA|$.

(e) Compute $|A^3|$ and $|B^8|$.

(f) Compute $|A^{-1}|$ and $|B^{-1}|$.