ADJOINTS, CRAMER'S RULE, APPLICATIONS IN GEOMETRY [LARSON 3.4]

- **ADJOINT OF A MATRIX:**

The **adjoint** of $n \times n$ square matrix A is defined as:

$$\operatorname{adj}(A) := \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}$$

i.e. The adjoint of A is the tranpose of the matrix of cofactors of A.

- **FINDING AN INVERSE OF A MATRIX VIA ITS ADJOINT:**

If A is $n \times n$ **invertible**, then $A^{-1} = \dfrac{1}{\det(A)}\operatorname{adj}(A)$.

Since all **cofactors** of A are necessary to find $\operatorname{adj}(A)$, use a **cofactor expansion** to quickly find $\det(A)$ afterwards.

- **THE VALUE OF FINDING AN INVERSE VIA ITS ADJOINT:**

Finding the inverse of a 3×3 or larger matrix via its adjoint is most useful when the entries of the matrix are **scalar functions** instead of just scalars:

$$\begin{bmatrix} e^{-t} & e^{2t} & 6e^{3t} \\ -e^{t} & 2e^{-t} & 5e^{4t} \\ -e^{6t} & -3e^{-3t} & 8e^{t} \end{bmatrix}, \quad \begin{bmatrix} (2-x-x^3) & (8+3x^2+4x^3) & (x-x^3) \\ (1-x-x^2) & (7-7x^2) & (x^2-x^3) \\ (4x-3x^2) & (9-x^3) & (1-x^3) \end{bmatrix}, \ldots.$$

Attempting to augment such matrices with the identity matrix and performing Gauss-Jordan Elimination would be extremely tedious & messy!!

- **CRAMER'S RULE:**

Given a square $n \times n$ linear system $A\mathbf{x} = \mathbf{b}$ with a **unique** solution. Then:

$$x_1 = \frac{\det(A_1)}{\det(A)}, \quad x_2 = \frac{\det(A_2)}{\det(A)}, \quad \ldots, \quad x_n = \frac{\det(A_n)}{\det(A)}$$

where the k^{th} column of A_k is the column vector $\mathbf{b}$.

- **AREA OF A TRIANGLE VIA A DETERMINANT:**

The area of a triangle with vertices $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ is

$$\text{Area } = \pm\frac{1}{2}\det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}$$

where the sign $(\pm)$ is chosen to ensure the area is **positive**.

- **COLLINEARITY OF 3 POINTS IN THE xy-PLANE:**

Let points $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$ and $P_3 = (x_3, y_3)$. Then:

$$\text{Points } P_1, P_2, P_3 \text{ are } \textbf{collinear} \iff \det\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} = 0$$

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EX 3.4.1: Using Cramer's Rule, solve the linear system: $\begin{cases} x_1 + x_2 + x_3 = 2 \\ 3x_1 + 5x_2 + 4x_3 = -1 \\ 3x_1 + 6x_2 + 5x_3 = 1 \end{cases}$

EX 3.4.2: Let points $P_1 = (4,5)$, $P_2 = (-2,-3)$, $P_3 = (0,6)$. Find the area of the triangle generated by P_1, P_2, P_3.

EX 3.4.3: Let points $P_1 = (5,19)$, $P_2 = (-4,-8)$, $P_3 = (13,43)$. Are P_1, P_2, P_3 collinear? (use a determinant)

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