EX 4.2.2: Show that $S := \left\{ \begin{bmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 1 \end{bmatrix} : a_{12}, a_{13}, a_{23} \in \mathbb{R} \right\}$ is <u>not</u> a vector space.

Three ways are shown below, but remember only <u>one</u> is sufficient to show S is not a vector space.

<u>METHOD 1:</u> Show that the 3×3 zero matrix is not in S :

$$O_{3\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies \text{Main diagonal entries of } O_{3\times 3} \text{ are not all ones} \implies O_{3\times 3} \notin S \implies \text{Zero Matrix is not contained}$$

<u>METHOD 2:</u> Show that some cleverly-chosen matrix is in S but its additive inverse is not in S :

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \implies -A = \begin{bmatrix} -1 & -2 & -2 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{bmatrix} \implies A \in S \text{ and } -A \notin S \implies \text{Additive Inverse is not closed}$$

<u>METHOD 3:</u> Show that some cleverly-chosen matrices are in S but their sum is not in S :

$$\text{Let } A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \implies A + B = \begin{bmatrix} 2 & 2 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{bmatrix} \implies A, B \in S \text{ and } A + B \notin S$$

$\implies$ Matrix Addition is not closed

<u>METHOD 4:</u> Show that some cleverly-chosen matrix is in S but some clever-chosen scalar multiple is not in S :

$$\text{Let } B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \implies 2B = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 2 & 6 \\ 0 & 0 & 2 \end{bmatrix} \implies B \in S \text{ and } 2B \notin S \implies \text{Scalar Multiplication is not closed}$$

EX 4.2.3: Show that $S := \left\{ \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{bmatrix} : a_{12}, a_{13}, a_{23} \geq 0 \right\}$ is <u>not</u> a vector space.

Notice that the matrix vector <u>is</u> in S : $O_{3\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in S$ (since all entries above main diagonal are ≥ 0)

Moreover, matrix addition <u>is</u> indeed closed in S:

$$\text{Let } A = \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & b_{12} & b_{13} \\ 0 & 0 & b_{23} \\ 0 & 0 & 0 \end{bmatrix} \text{ such that } \begin{matrix} a_{12}, a_{13}, a_{23} \geq 0 \\ \text{and} \\ b_{12}, b_{13}, b_{23} \geq 0 \end{matrix} \quad \text{Then, } A, B \in S$$

$$\text{Moreover, } A + B = \begin{bmatrix} 0 & (a_{12} + b_{12}) & (a_{13} + b_{13}) \\ 0 & 0 & (a_{23} + b_{23}) \\ 0 & 0 & 0 \end{bmatrix} \text{ and } \begin{matrix} a_{12} + b_{12} \geq 0 \\ a_{13} + b_{13} \geq 0 \\ a_{23} + b_{23} \geq 0 \end{matrix} \implies A + B \in S \;\therefore\; \text{Matrix Addition is closed}$$

Therefore, one of the below ways is necessary to show S is not a vector space.

<u>METHOD 1:</u> Show that some cleverly-chosen matrix is in S but its additive inverse is not in S :

$$\text{Let } A = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \implies -A = \begin{bmatrix} 0 & -2 & -2 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \implies A \in S \text{ and } -A \notin S \implies \text{Additive Inverse is not closed}$$

<u>METHOD 2:</u> Show that some cleverly-chosen matrix is in S but some clever-chosen scalar multiple is not in S :

$$\text{Let } B = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \implies (-2)B = \begin{bmatrix} 0 & -4 & -6 \\ 0 & 0 & -6 \\ 0 & 0 & 0 \end{bmatrix} \implies B \in S \text{ and } (-2)B \notin S \implies \text{Scalar Multiplication is not closed}$$

EX 4.2.4: Show that $S := \{at^2 + bt + c : a \text{ is divisible by 2 and } b, c \in \mathbb{R}\}$ is <u>not</u> a vector space.

Notice that the zero quadratic <u>is</u> in S : $z(t) = 0t^2 + 0t + 0 \implies \begin{matrix} a = 0 \\ b = 0 \\ c = 0 \end{matrix} \implies \begin{matrix} a \text{ is divisible by 2} \\ b \in \mathbb{R} \\ c \in \mathbb{R} \end{matrix} \implies z(t) \in S$

Moreover, every additive inverse <u>is</u> in S :

Let $p(t) = at^2 + bt + c$ such that $\begin{matrix} a \text{ is divisible by 2} \\ b \in \mathbb{R} \\ c \in \mathbb{R} \end{matrix}$ Then, $p(t) \in S$

Hence, $-p(t) = (-a)t^2 + (-b)t + (-c) \implies \begin{matrix} -a \text{ is divisible by 2} \\ -b \in \mathbb{R} \\ -c \in \mathbb{R} \end{matrix} \implies \begin{matrix} -p(t) \in S \\ \therefore \text{ Additive inverse is closed} \end{matrix}$

Further still, every polynomial sum <u>is</u> in S :

Let $\begin{matrix} p(t) = a_1t^2 + b_1t + c_1 \\ q(t) = a_2t^2 + b_2t + c_2 \end{matrix}$ such that $\begin{matrix} a_1, a_2 \text{ are each divisible by 2} \\ b_1, b_2 \in \mathbb{R} \\ c_1, c_2 \in \mathbb{R} \end{matrix}$ Then, $p(t), q(t) \in S$

Hence, $p(t)+q(t) = (a_1+a_2)t^2+(b_1+b_2)t+(c_1+c_2) \implies \begin{matrix} (a_1 + a_2) \text{ is divisible by 2} \\ (b_1 + b_2) \in \mathbb{R} \\ (c_1 + c_2) \in \mathbb{R} \end{matrix} \implies \begin{matrix} p(t) + q(t) \in S \\ \therefore \text{ Polynomial Addition is closed} \end{matrix}$

Therefore, the only way that works is to show scalar multiplication is not closed.

<u>METHOD 1:</u> Show that some cleverly-chosen quadratic is in S but some cleverly-chosen scalar multiple is not in S :

Let $q(t) = 2t^2+2t+2 \implies \left(\frac{\pi}{2}\right) q(t) = \pi t^2+\pi t+\pi \implies q(t) \in S$ and $\left(\frac{\pi}{2}\right) q(t) \notin S \implies$ Scalar Multiplication is not closed

(Note that $\left(\frac{\pi}{2}\right) q(t) \notin S$ since π is not divisible by 2)

(Notice that $2q(t) \in S$, $3q(t) \in S$, etc... This illustrates that sometimes a **clever** choice of the scalar is critical.)