VECTOR SPACES [LARSON 4.2]

- **VECTOR SPACE (DEFINITION):** Let V be a set on which vector addition and scalar mult. are well-defined.

  Then V is a **vector space** if these axioms hold $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ and $\forall \alpha, \beta \in \mathbb{R}$:

$\mathbf{u} + \mathbf{v} \in V$	(Closure under Addition)
$\alpha \mathbf{v} \in V$	(Closure under Scalar Multiplication)
$\vec{\mathbf{0}} \in V$	(Containment of the Zero Vector)
$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	(Commutativity of Vector Addition)
$\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$	(Associativity of Vector Addition)
$\mathbf{v} + \vec{\mathbf{0}} = \mathbf{v}$	(Additive Identity = Zero Vector)
$\mathbf{v} + (-\mathbf{v}) = \vec{\mathbf{0}}$	(Vector + Its Additive Inverse = Additive Identity)
$\alpha(\mathbf{u} + \mathbf{v}) = \alpha \mathbf{u} + \alpha \mathbf{v}$	(Scalar Mult. Distributes over Vector Addition)
$(\alpha + \beta)\mathbf{v} = \alpha \mathbf{v} + \beta \mathbf{v}$	(Scalar Mult. Distributes over Scalar Addition)
$\alpha(\beta \mathbf{v}) = (\alpha\beta)\mathbf{v}$	(Associativity of Scalar Multiplication)
$1(\mathbf{v}) = \mathbf{v}$	(Scalar Multiplicative Identity)

- **COMMON VECTOR SPACES:**

$\mathbb{R}$	$\equiv$	Set of all real numbers (scalars)
$\mathbb{R}^n$	$\equiv$	Set of all ordered n-tuples (n-wide vectors)
$\mathbb{R}^{m \times n}$	$\equiv$	Set of all $m \times n$ matrices
P_n	$\equiv$	Set of all polynomials of degree n or less
$C[a, b]$	$\equiv$	Set of all continuous functions on $[a, b]$
$C^1[a, b]$	$\equiv$	Set of all differentiable functions on $[a, b]$
$C^2[a, b]$	$\equiv$	Set of all twice-differentiable functions on $[a, b]$
$C(-\infty, \infty)$	$\equiv$	Set of all continuous functions on $(-\infty, \infty)$ [i.e. continuous everywhere]

  REMARK: Always assume that the operations of vector addition & scalar multiplication are the standard definitions.

- **EXAMPLE VECTORS & ZERO VECTORS IN COMMON VECTOR SPACES:**

VECTOR SPACE	"VECTOR" LABELS	EXAMPLE "VECTORS"	"ZERO VECTOR"
$\mathbb{R}$	a, b, c	Scalars: $1, -3/2, \sqrt{2}, \pi$	0
$\mathbb{R}^2$	$\mathbf{u}, \mathbf{v}, \mathbf{w}$	2-Wide Vectors: $(1, 1), (-3, 4), (\sqrt{2}, \pi)$	$\vec{\mathbf{0}} = (0, 0)$
$\mathbb{R}^3$	$\mathbf{u}, \mathbf{v}, \mathbf{w}$	3-Wide Vectors: $(1, 1, 1), (\sqrt{2}, \pi, -1)$	$\vec{\mathbf{0}} = (0, 0, 0)$
$\mathbb{R}^{3 \times 2}$	A, B, C	3×2 Matrices: $\begin{bmatrix} 1 & 2 \\ -3 & \sqrt{5} \\ -\pi & 1/6 \end{bmatrix}$	$O_{3 \times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$
$\mathbb{R}^{2 \times 2}$	A, B, C	2×2 Matrices: $\begin{bmatrix} 1 & 2 \\ -3 & \sqrt{5} \end{bmatrix}$	$O_{2 \times 2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
P_1	p, q, r	Polynomials: $3, 4 - 2t$	$z(t) = 0 + 0t$
P_2	p, q, r	Polynomials: $3, 4 - 2t, 5 + t - 7t^2$	$z(t) = 0 + 0t + 0t^2$
P_3	p, q, r	Polynomials: $3 - 4t + 2t^2 + 5t^3$	$z(t) = 0 + 0t + 0t^2 + 0t^3$
$C[0, 1]$	f, g, h	Functions: $x^2, \sin x, \sqrt{1 + x}, \frac{1}{x-2}$	$z(x) = 0$ on $[0, 1]$
$C(-\infty, \infty)$	f, g, h	Functions: $x^2, \sin x, e^x, \sqrt[3]{x}, \lvert x \rvert$	$z(x) = 0$
$C^1(-\infty, \infty)$	f, g, h	Functions: $x^2, \sin x, e^x$	$z(x) = 0$

- **SHOWING THAT A SET IS NOT A VECTOR SPACE:** ($\exists \equiv$ "There exists at least one ...")

  A set S is **not** a vector space if at least one of the following is true:

  - The zero vector is not in the set: $\vec{\mathbf{0}} \notin S$
  - The additive inverse is not in the set: $\mathbf{v} \in S$ and $-\mathbf{v} \notin S$
  - Closure of Vector Addition fails: $\exists \mathbf{u}, \mathbf{v} \in S$ such that $\mathbf{u} + \mathbf{v} \notin S$
  - Closure of Scalar Multiplication fails: $\exists \mathbf{v} \in S, \alpha \in \mathbb{R}$ such that $\alpha \mathbf{v} \notin S$

EX 4.2.1: Show that $S := \left\{ \begin{bmatrix} x_1 \\ x_1 \\ 1+x_1 \end{bmatrix} : x_1 \in \mathbb{R} \right\}$ is not a vector space.

EX 4.2.2: Show that $S := \left\{ \begin{bmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 1 \end{bmatrix} : a_{12}, a_{13}, a_{23} \in \mathbb{R} \right\}$ is not a vector space.

EX 4.2.3: Show that $S := \left\{ \begin{bmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{bmatrix} : a_{12}, a_{13}, a_{23} \geq 0 \right\}$ is not a vector space.

EX 4.2.4: Show that $S := \{at^2 + bt + c : a \text{ is divisible by 2 and } b, c \in \mathbb{R}\}$ is not a vector space.