EX 4.3.1: Show: $W = \{p(t) \in P_3 : p'(1) = 0\}$ <u>is</u> a subspace of P_3.

Clearly, $W \subseteq P_3$ (since every element of W is also an element of P_3)

Observe that the zero cubic polynomial $z(t) = 0t^3 + 0t^2 + 0t + 0 \in W$ since $z'(1) = 0$

Let $p(t), q(t) \in W$ and $\alpha \in \mathbb{R}$. Then $p'(1) = 0$ and $q'(1) = 0$.

$\implies [p+q]'(1) = p'(1) + q'(1) = 0 + 0 = 0 \implies p(t) + q(t) \in W \implies$ Polynomial Addition is closed in W

$\implies [\alpha p]'(1) = \alpha p'(1) = \alpha(0) = 0 \implies \alpha p(t) \in W \implies$ Scalar Multiplication is closed in W

Therefore, W is a subspace of P_3.

EX 4.3.2: Show that the set of all 3×3 lower triangular matrices <u>is</u> a subspace of $\mathbb{R}^{3\times3}$.

Let W be the set of all 3×3 lower triangular matrices. Then $W = \left\{ \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} : a_{11}, \dots, a_{33} \in \mathbb{R} \right\}$

Clearly, $W \subseteq \mathbb{R}^{3\times3}$ (since every element of W is also an element of $\mathbb{R}^{3\times3}$)

Observe that the zero matrix $O_{3\times3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in W$ since it's 3×3 lower triangular

Let $A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$, $B = \begin{bmatrix} b_{11} & 0 & 0 \\ b_{21} & b_{22} & 0 \\ b_{31} & b_{32} & b_{33} \end{bmatrix}$ and $\alpha \in \mathbb{R}$. Then $A, B \in W$.

$\implies A + B = \begin{bmatrix} (a_{11}+b_{11}) & 0 & 0 \\ (a_{21}+b_{21}) & (a_{22}+b_{22}) & 0 \\ (a_{31}+b_{31}) & (a_{32}+b_{32}) & (a_{33}+b_{33}) \end{bmatrix} \implies A + B \in W \implies$ Matrix Addition is closed in W

$\implies \alpha B = \begin{bmatrix} \alpha b_{11} & 0 & 0 \\ \alpha b_{21} & \alpha b_{22} & 0 \\ \alpha b_{31} & \alpha b_{32} & \alpha b_{33} \end{bmatrix} \implies \alpha B \in W \implies$ Scalar Multiplication is closed in W

Therefore, W is a subspace of $\mathbb{R}^{3\times3}$.

EX 4.3.3: Show that the set of all 3×3 **<u>unit</u>** lower triangular matrices is <u>not</u> a subspace of $\mathbb{R}^{3\times3}$.

Let W be the set of all 3×3 **<u>unit</u>** lower triangular matrices. Then $W = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ a_{21} & 1 & 0 \\ a_{31} & a_{32} & 1 \end{bmatrix} : a_{21}, a_{31}, a_{32} \in \mathbb{R} \right\}$

Clearly, $W \subseteq \mathbb{R}^{3\times3}$ (since every element of W is also an element of $\mathbb{R}^{3\times3}$)

Three ways are shown below, but remember only <u>one</u> is sufficient to show W is not a subspace.

<u>METHOD 1:</u> Show that the 3×3 zero matrix is not in W :

$$O_{3\times3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \implies \text{Main diagonal entries of } O_{3\times3} \text{ are not all ones} \implies O_{3\times3} \notin W$$

<u>METHOD 2:</u> Show that matrix addition is not closed in W :

$$\text{Let } A = \begin{bmatrix} 1 & 0 & 0 \\ a_{21} & 1 & 0 \\ a_{31} & a_{32} & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & 0 \\ b_{21} & 1 & 0 \\ b_{31} & b_{32} & 1 \end{bmatrix} \implies A + B = \begin{bmatrix} 2 & 0 & 0 \\ (a_{21}+b_{21}) & 2 & 0 \\ (a_{31}+b_{31}) & (a_{32}+b_{32}) & 2 \end{bmatrix} \notin W$$

<u>METHOD 3:</u> Show that scalar multiplication is not closed in W :

$$\text{Let } B = \begin{bmatrix} 1 & 0 & 0 \\ b_{21} & 1 & 0 \\ b_{31} & b_{32} & 1 \end{bmatrix} \text{ and } \alpha \neq 1 \implies \alpha B = \begin{bmatrix} \alpha & 0 & 0 \\ \alpha b_{21} & \alpha & 0 \\ \alpha b_{31} & \alpha b_{32} & \alpha \end{bmatrix} \notin W \text{ (since main diagonal entries } \neq 1)$$

EX 4.3.4: Show: $W = \left\{ \begin{bmatrix} x_1 \\ 0 \end{bmatrix} : x_1 \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^2$.

Clearly, $W \subseteq \mathbb{R}^2$ (since every element of W is also an element of $\mathbb{R}^2$)

Observe that the zero vector $\vec{\mathbf{0}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in W$ since 2^{nd} component of $\vec{\mathbf{0}}$ is zero

Let $\mathbf{u}, \mathbf{v} \in W$ and $\alpha \in \mathbb{R}$. Then $\mathbf{u} = \begin{bmatrix} u_1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ 0 \end{bmatrix}$ where $u_1, v_1 \in \mathbb{R}$

$\implies \mathbf{u} + \mathbf{v} = \begin{bmatrix} (u_1 + v_1) \\ 0 \end{bmatrix} \implies \mathbf{u} + \mathbf{v} \in W \implies$ Vector Addition is closed in W

$\implies \alpha\mathbf{v} = \begin{bmatrix} \alpha v_1 \\ 0 \end{bmatrix} \implies \alpha\mathbf{v} \in W \implies$ Scalar Multiplication is closed in W

Therefore, W is a subspace of $\mathbb{R}^2$.

EX 4.3.5: Show: $W = \left\{ \begin{bmatrix} x_1 \\ 1 \end{bmatrix} : x_1 \in \mathbb{R} \right\}$ is not a subspace of $\mathbb{R}^2$.

Clearly, $W \subseteq \mathbb{R}^2$ (since every element of W is also an element of $\mathbb{R}^2$)

Three ways are shown below, but remember only one is sufficient to show W is not a subspace.

METHOD 1: Show that the zero vector is not in W :

$\vec{\mathbf{0}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies 2^{nd}$ component of $\vec{\mathbf{0}}$ is not one $\implies \vec{\mathbf{0}} \notin W$

METHOD 2: Show that vector addition is not closed in W :

Let $\mathbf{u} = \begin{bmatrix} u_1 \\ 1 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} v_1 \\ 1 \end{bmatrix} \implies \mathbf{u} + \mathbf{v} = \begin{bmatrix} (u_1 + v_1) \\ 2 \end{bmatrix} \notin W$ (since 2^{nd} component $\neq 1$)

METHOD 3: Show that scalar multiplication is not closed in W :

Let $\mathbf{v} = \begin{bmatrix} v_1 \\ 1 \end{bmatrix}$ and $\alpha \neq 1 \implies \alpha\mathbf{v} = \begin{bmatrix} \alpha v_1 \\ \alpha \end{bmatrix} \notin W$ (since 2^{nd} component $\neq 1$)

EX 4.3.6: Show: $W = \{p(t) \in P_3 : \int_1^2 p(t)\, dt = 3\}$ is not a subspace of P_3.

Clearly, $W \subseteq P_3$ (since every element of W is also an element of P_3)

Three ways are shown below, but remember only one is sufficient to show W is not a subspace.

METHOD 1: Show that the zero cubic polynomial is not in W : (Here, $k \in \mathbb{R}$ is an arbitrary constant of integration.)

$$z(t) = 0t^3 + 0t^2 + 0t + 0 \implies \int_1^2 z(t)\, dt = \int_1^2 0\, dt = \Big[k\Big]_{t=1}^{t=2} \stackrel{FTC}{=} k - k = 0 \implies \int_1^2 z(t)\, dt \neq 3 \implies z(t) \notin W$$

METHOD 2: Show that polynomial addition is not closed in W :

Let $p(t), q(t) \in W$. Then $\int_1^2 p(t)\, dt = 3$ and $\int_1^2 q(t)\, dt = 3$

$$\implies \int_1^2 [p(t) + q(t)]\, dt = \int_1^2 p(t)\, dt + \int_1^2 q(t)\, dt = 3 + 3 = 6$$

$$\implies \int_1^2 [p(t) + q(t)]\, dt \neq 3 \implies p(t) + q(t) \notin W$$

METHOD 3: Show that scalar multiplication is not closed in W :

Let $q(t) \in W$ and $\alpha \neq 1$. Then $\int_1^2 q(t)\, dt = 3$

$$\implies \int_1^2 \alpha q(t)\, dt = \alpha \int_1^2 q(t)\, dt = \alpha(3) = 3\alpha \neq 3$$

$$\implies \int_1^2 \alpha q(t)\, dt \neq 3 \implies \alpha q(t) \notin W$$