SUBSPACES [LARSON 4.3]

- **SUBSPACE OF A VECTOR SPACE (DEFINITION):**

  Let V be a vector space.

  Then a nonempty set W is a **subspace** of V if the following all hold:

$W \subseteq V$	(W is a **subset** of V)
$\vec{\mathbf{0}} \in W$	(W contains the **zero vector**)
$\mathbf{u}, \mathbf{v} \in W \implies \mathbf{u} + \mathbf{v} \in W$	(W is **closed under vector addition**)
$\mathbf{v} \in W, \alpha \in \mathbb{R} \implies \alpha\mathbf{v} \in W$	(W is **closed under scalar multiplication**)

- **TRIVIAL SUBSPACES OF A VECTOR SPACE:**

  Let V be a vector space. Then $\{\vec{\mathbf{0}}\}$ and V are the two **trivial subspaces** of V.

  REMARK: $\{\vec{\mathbf{0}}\}$ is sometimes called the **zero subspace**.

- **THE INTERSECTION OF TWO SUBSPACES IS A SUBSPACE:**

  Let W_1, W_2 both be subspaces of vector space V. Then, $W_1 \cap W_2$ is a subspace of V.

- **ESTABLISHING THAT A SET IS NOT A SUBSPACE:**

  W is **not** a subspace of vector space V if at least one of the following is true:

  - W is not a subset of V: $W \not\subseteq V$
  - The zero vector is not in W: $\vec{\mathbf{0}} \notin W$
  - Closure of Vector Addition fails: $\exists \mathbf{u}, \mathbf{v} \in W$ such that $\mathbf{u} + \mathbf{v} \notin W$
  - Closure of Scalar Multiplication fails: $\exists \mathbf{v} \in W, \alpha \in \mathbb{R}$ such that $\alpha\mathbf{v} \notin W$

- **LINEARITY OF DIFFERENTIATION (REVIEW FROM CALCULUS):**

  Let $f, g \in C^k[a,b]$ where $k \geq 1$ is an integer and $\alpha \in \mathbb{R}$. Then:

$$[f+g]'(x) \equiv \frac{d}{dx}\Big[f(x)+g(x)\Big] = \frac{d}{dx}\Big[f(x)\Big] + \frac{d}{dx}\Big[g(x)\Big] \equiv f'(x) + g'(x)$$

$$[\alpha f]'(x) \equiv \frac{d}{dx}\Big[\alpha f(x)\Big] = \alpha\frac{d}{dx}\Big[f(x)\Big] \equiv \alpha f'(x)$$

$$[f+g]^{(k)}(x) \equiv \frac{d^k}{dx^k}\Big[f(x)+g(x)\Big] = \frac{d^k}{dx^k}\Big[f(x)\Big] + \frac{d^k}{dx^k}\Big[g(x)\Big] \equiv f^{(k)}(x) + g^{(k)}(x)$$

$$[\alpha f]^{(k)}(x) \equiv \frac{d^k}{dx^k}\Big[\alpha f(x)\Big] = \alpha\frac{d^k}{dx^k}\Big[f(x)\Big] \equiv \alpha f^{(k)}(x)$$

- **LINEARITY OF INTEGRATION (REVIEW FROM CALCULUS):**

  Let $f, g \in C[a,b]$ and $\alpha \in \mathbb{R}$. Then:

$$\int_a^b \Big[f(x)+g(x)\Big]\, dx = \int_a^b f(x)\, dx + \int_a^b g(x)\, dx$$

$$\int_a^b \Big[\alpha f(x)\Big]\, dx = \alpha\int_a^b f(x)\, dx$$

EX 4.3.1: Show: $W = \{p(t) \in P_3 : p'(1) = 0\}$ is a subspace of P_3.

EX 4.3.2: Show that the set of all 3×3 lower triangular matrices is a subspace of $\mathbb{R}^{3\times 3}$.

EX 4.3.3: Show that the set of all 3×3 **unit** lower triangular matrices is not a subspace of $\mathbb{R}^{3\times 3}$.

EX 4.3.4: Show: $W = \left\{ \begin{bmatrix} x_1 \\ 0 \end{bmatrix} : x_1 \in \mathbb{R} \right\}$ <u>is</u> a subspace of $\mathbb{R}^2$.

EX 4.3.5: Show: $W = \left\{ \begin{bmatrix} x_1 \\ 1 \end{bmatrix} : x_1 \in \mathbb{R} \right\}$ is <u>not</u> a subspace of $\mathbb{R}^2$.

EX 4.3.6: Show: $W = \{p(t) \in P_3 : \int_1^2 p(t)\ dt = 3\}$ is <u>not</u> a subspace of P_3.