EX 4.4.3: Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $S = \left\{ \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ -9 \\ 9 \end{bmatrix} \right\} \equiv \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \subseteq \mathbb{R}^3$.

(a) Write $\mathbf{u}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, if possible. (b) Is $\mathbf{u} \in \text{span}(S)$?

$$[A \mid \mathbf{u}] = \left[\begin{array}{ccc|c} | & | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 & \mathbf{u} \\ | & | & | & | \end{array}\right] = \left[\begin{array}{ccc|c} \boxed{1} & 3 & -3 & 1 \\ 3 & 2 & -9 & 2 \\ -3 & 3 & 9 & 3 \end{array}\right] \xrightarrow[3R_1+R_3\to R_3]{(-3)R_1+R_2\to R_2} \left[\begin{array}{ccc|c} \boxed{1} & 3 & -3 & 1 \\ 0 & -7 & 0 & -1 \\ 0 & 12 & 0 & 6 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} \boxed{1} & 3 & -3 & 1 \\ 0 & 12 & 0 & 6 \\ 0 & -7 & 0 & -1 \end{array}\right]$$

$$\xrightarrow{(\frac{1}{12})R_2\to R_2} \left[\begin{array}{ccc|c} \boxed{1} & 3 & -3 & 1 \\ 0 & \boxed{1} & 0 & 1/2 \\ 0 & -7 & 0 & -1 \end{array}\right] \xrightarrow{7R_2+R_3\to R_3} \left[\begin{array}{ccc|c} \boxed{1} & 3 & -3 & 1 \\ 0 & \boxed{1} & 0 & 1/2 \\ 0 & 0 & 0 & 5/2 \end{array}\right] \implies 0 = 5/2 \leftarrow \text{CONTRADICTION!}$$

$\therefore$ (a) $\boxed{\text{Desired linear combination is not possible}}$

(b) $\boxed{\text{No, } \mathbf{u} \notin \text{span}(S)}$ since $\mathbf{u}$ is not a linear combination of the vectors in S

EX 4.4.4: Let $r(t) = 3t + 2$ and $S = \{2t^2 - 3t + 6,\ 4t^2 + t,\ t^2 + 5t - 1\} \equiv \{p_1(t), p_2(t), p_3(t)\} \subseteq P_2$.

(a) Write $r(t)$ as a linear combination of $p_1(t), p_2(t), p_3(t)$, if possible. (b) Is $r(t) \in \text{span}(S)$?

$$\left[\begin{array}{ccc|c} | & | & | & | \\ p_1(t) & p_2(t) & p_3(t) & r(t) \\ | & | & | & | \end{array}\right] = \left[\begin{array}{ccc|c} 2 & 4 & 1 & 0 \\ -3 & 1 & 5 & 3 \\ 6 & 0 & -1 & 2 \end{array}\right] \xrightarrow[2R_2\to R_2]{3R_1\to R_1} \left[\begin{array}{ccc|c} 6 & 12 & 3 & 0 \\ -6 & 2 & 10 & 6 \\ 6 & 0 & -1 & 2 \end{array}\right] \xrightarrow[(-1)R_1+R_3\to R_3]{R_1+R_2\to R_2} \left[\begin{array}{ccc|c} 6 & 12 & 3 & 0 \\ 0 & 14 & 13 & 6 \\ 0 & -12 & -4 & 2 \end{array}\right]$$

$$\xrightarrow[(\frac{1}{2})R_2\to R_2]{(\frac{1}{6})R_1\to R_1} \left[\begin{array}{ccc|c} \boxed{1} & 2 & 1/2 & 0 \\ 0 & 7 & 13/2 & 3 \\ 0 & -6 & -2 & 1 \end{array}\right] \xrightarrow{R_3+R_2\to R_2} \left[\begin{array}{ccc|c} \boxed{1} & 2 & 1/2 & 0 \\ 0 & \boxed{1} & 9/2 & 4 \\ 0 & -6 & -2 & 1 \end{array}\right] \xrightarrow{6R_2+R_3\to R_3} \left[\begin{array}{ccc|c} \boxed{1} & 2 & 1/2 & 0 \\ 0 & \boxed{1} & 9/2 & 4 \\ 0 & 0 & 25 & 25 \end{array}\right]$$

$$\xrightarrow{(\frac{1}{25})R_3\to R_3} \left[\begin{array}{ccc|c} \boxed{1} & 2 & 1/2 & 0 \\ 0 & \boxed{1} & 9/2 & 4 \\ 0 & 0 & \boxed{1} & 1 \end{array}\right] \xrightarrow[(-\frac{1}{2})R_3+R_1\to R_1]{(-\frac{9}{2})R_3+R_2\to R_2} \left[\begin{array}{ccc|c} \boxed{1} & 2 & 0 & -1/2 \\ 0 & \boxed{1} & 0 & -1/2 \\ 0 & 0 & \boxed{1} & 1 \end{array}\right] \xrightarrow{(-2)R_2+R_1\to R_1} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 1/2 \\ 0 & \boxed{1} & 0 & -1/2 \\ 0 & 0 & \boxed{1} & 1 \end{array}\right]$$

$\therefore$ (a) $\boxed{r(t) = \frac{1}{2}p_1(t) - \frac{1}{2}p_2(t) + p_3(t)}$

(b) $\boxed{\text{Yes, } r(t) \in \text{span}(S)}$ since $r(t)$ is a linear combination of the polynomials in S

EX 4.4.5: Let $r(t) = -8t^2 - 16t + 24$ and $S = \{t^2 + 2t - 3,\ -4t^2 - 8t + 12,\ 2t^2 + 4t - 6\} \equiv \{p_1(t), p_2(t), p_3(t)\} \subseteq P_2$.

(a) Write $r(t)$ as a linear combination of $p_1(t), p_2(t), p_3(t)$, if possible. (b) Is $r(t) \in \text{span}(S)$?

$$[A \mid r(t)] = \left[\begin{array}{ccc|c} \boxed{1} & -4 & 2 & -8 \\ 2 & -8 & 4 & -16 \\ -3 & 12 & -6 & 24 \end{array}\right] \xrightarrow[3R_1+R_3\to R_3]{(-2)R_1+R_2\to R_2} \left[\begin{array}{ccc|c} \boxed{1} & -4 & 2 & -8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]. \text{ Let } \begin{cases} c_2 = s \\ c_3 = \widetilde{t} \end{cases} \text{ Then: } \begin{array}{rl} & c_1 - 4c_2 + 2c_3 = -8 \\ \implies & c_1 - 4s + 2\widetilde{t} = -8 \\ \implies & c_1 = -8 + 4s - 2\widetilde{t} \end{array}$$

$\therefore (c_1, c_2, c_3) = (-8 + 4s - 2\widetilde{t}, s, \widetilde{t})$ Pick convenient values for $s, \widetilde{t}$: Let $s = \widetilde{t} = 0$. Then $(c_1, c_2, c_3) = (-8, 0, 0)$

$\therefore$ (a) $\boxed{r(t) = (-8)p_1(t) + (0)p_2(t) + (0)p_3(t)}$

(b) $\boxed{\text{Yes, } r(t) \in \text{span}(S)}$ since $r(t)$ is a linear combination of the polynomials in S

EX 4.4.6: Let $r(t) = 3t^2 + t + 5$ and $S = \{t^2 + 2t - 3,\ -4t^2 - 8t + 12,\ 2t^2 + 4t - 6\} \equiv \{p_1(t), p_2(t), p_3(t)\} \subseteq P_2$.

(a) Write $r(t)$ as a linear combination of $p_1(t), p_2(t), p_3(t)$, if possible. (b) Is $r(t) \in \text{span}(S)$?

$$[A \mid r(t)] = \left[\begin{array}{ccc|c} \boxed{1} & -4 & 2 & 3 \\ 2 & -8 & 4 & 1 \\ -3 & 12 & -6 & 5 \end{array}\right] \xrightarrow[3R_1+R_3\to R_3]{(-2)R_1+R_2\to R_2} \left[\begin{array}{ccc|c} \boxed{1} & -4 & 2 & 3 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & 0 & 14 \end{array}\right] \begin{array}{l} \implies 0 = -5 \leftarrow \text{CONTRADICTION!} \\ \implies 0 = 14 \leftarrow \text{CONTRADICTION!} \end{array}$$

$\therefore$ (a) $\boxed{\text{Desired linear combination is not possible}}$

(b) $\boxed{\text{No, } r(t) \notin \text{span}(S)}$ since $r(t)$ is not a linear combination of the polynomials in S