EX 4.4.10: Let $S = \{2t^2 - 3t + 6,\ 4t^2 + t,\ t^2 + 5t - 1\} \equiv \{p_1(t), p_2(t), p_3(t)\} \subseteq P_2$.

(a) Does S span P_2? (b) Is S linearly independent or dependent?

$$A = \begin{bmatrix} | & | & | \\ p_1(t) & p_2(t) & p_3(t) \\ | & | & | \end{bmatrix} = \begin{bmatrix} 2 & 4 & 1 \\ -3 & 1 & 5 \\ 6 & 0 & -1 \end{bmatrix} \xrightarrow[2R_2 \to R_2]{3R_1 \to R_1} \begin{bmatrix} 6 & 12 & 3 \\ -6 & 2 & 10 \\ 6 & 0 & -1 \end{bmatrix} \xrightarrow[(-1)R_1 + R_3 \to R_3]{R_1 + R_2 \to R_2} \begin{bmatrix} 6 & 12 & 3 \\ 0 & 14 & 13 \\ 0 & -12 & -4 \end{bmatrix}$$

$$\xrightarrow[(-\frac{1}{4})R_3 \to R_3]{(\frac{1}{3})R_1 \to R_1} \begin{bmatrix} 2 & 4 & 1 \\ 0 & 14 & 13 \\ 0 & 3 & 1 \end{bmatrix} \xrightarrow[14R_3 \to R_3]{3R_2 \to R_2} \begin{bmatrix} 2 & 4 & 1 \\ 0 & 42 & 39 \\ 0 & 42 & 14 \end{bmatrix} \xrightarrow{(-1)R_2 + R_3 \to R_3} \begin{bmatrix} 2 & 4 & 1 \\ 0 & 42 & 39 \\ 0 & 0 & -25 \end{bmatrix} \xrightarrow[(-\frac{1}{25})R_3 \to R_3]{(\frac{1}{3})R_2 \to R_2} \begin{bmatrix} 2 & 4 & 1 \\ 0 & 14 & 13 \\ 0 & 0 & \boxed{1} \end{bmatrix}$$

$$\xrightarrow[(-1)R_3 + R_1 \to R_1]{(-13)R_3 + R_2 \to R_2} \begin{bmatrix} 2 & 4 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix} \xrightarrow[(\frac{1}{14})R_2 \to R_2]{(\frac{1}{2})R_1 \to R_1} \begin{bmatrix} \boxed{1} & 2 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix} \xrightarrow{(-2)R_2 + R_1 \to R_1} \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix} = \text{RREF}(A)$$

(a) Since every row of RREF(A) has a pivot, $\boxed{\mathcal{S} \text{ spans } P_2.}$

(b) Since every column of RREF(A) has a pivot, $\boxed{\mathcal{S} \text{ is linearly independent.}}$

EX 4.4.11: Let $S = \{t^2 + 2t - 3,\ -4t^2 - 8t + 12,\ 2t^2 + 4t - 6\} \equiv \{p_1(t), p_2(t), p_3(t)\} \subseteq P_2$.

(a) Does S span P_2? (b) Is S linearly independent or dependent?

$$A = \begin{bmatrix} | & | & | \\ p_1(t) & p_2(t) & p_3(t) \\ | & | & | \end{bmatrix} = \begin{bmatrix} \boxed{1} & -4 & 2 \\ 2 & -8 & 4 \\ -3 & 12 & -6 \end{bmatrix} \xrightarrow[3R_1 + R_3 \to R_3]{(-2)R_1 + R_2 \to R_2} \begin{bmatrix} \boxed{1} & -4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{RREF}(A)$$

(a) Since at least one row of RREF(A) has no pivot, $\boxed{\mathcal{S} \text{ does } \underline{\text{not}} \text{ span } P_2.}$

(b) Since at least one column of RREF(A) has no pivot, $\boxed{\mathcal{S} \text{ is } \underline{\text{not}} \text{ linearly independent.}}$

Moreover:

Column 2 of RREF(A) implies that $p_2(t) = (-4)p_1(t)$

Column 3 of RREF(A) implies that $p_3(t) = 2p_1(t)$