# VECTOR SPACES: LINEAR COMBINATIONS, SPANNING SETS [LARSON 4.4]

- **LINEAR COMBINATIONS OF VECTORS:** Let $V$ be a vector space.

  Then a vector $\mathbf{u} \in V$ is represented as a **linear combination** of vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in V$ if $\mathbf{u}$ can be written as

  $$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k$$

  where scalars $c_1, c_2, \ldots, c_k \in \mathbb{R}$

- **FINDING A LINEAR COMBINATION (SIMPLIFIED PROCEDURE):**

  TASK: Write $\mathbf{u} \in V$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \in V$.

  (1) Let $\mathbf{c} = (c_1, c_2, \ldots, c_k) \subseteq \mathbb{R}^k$ be unknown scalars such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{u}$

  (2) Solve the linear system $A\mathbf{c} = \mathbf{u}$ for $\mathbf{c}$ using Gauss-Jordan on $[A \mid \mathbf{u}] = \left[\begin{array}{ccc|c} | & & | & | \\ \mathbf{v}_1 & \cdots & \mathbf{v}_k & \mathbf{u} \\ | & & | & | \end{array}\right]$

  ($\star$) If there are infinitely many solutions, let the parameters $t, s, \ldots$ be any values (e.g. Let $t = 1,\ s = 0,\ \ldots$)

  ($\star$) If there are no solutions, it's not possible to write $\mathbf{u}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$

  NOTE: For **polynomials**, form each column of matrix $A$ using the **coefficients** of each polynomial.

- **SPANNING SET OF A VECTOR SPACE:**

  Let $V$ be a vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\} \subseteq V$

  Then the **span of** $S$ is the **set of all linear combination** of vectors in $S$:

  $$\text{span}(S) \equiv \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\} := \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k : c_1, c_2, \cdots, c_k \in \mathbb{R}\}$$

  Moreover, $S$ **spans** $V$ if $\text{span}(S) = V$.

  i.e. $S$ **spans** $V$ if every vector of $V$ can be written as a linear combination of vectors in $S$.

- **A SPANNING SET IS A SUBSPACE:**

  Let $V$ be a vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\} \subseteq V$

  Then $\text{span}(S)$ is a subspace of $V$. Moreover, $\text{span}(S)$ is the smallest subspace of $V$ that contains $S$.

- **LINEAR INDEPENDENCE OF A SET OF VECTORS:** Let $V$ be a vector space and $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\} \subseteq V$

  Then $S$ is called **linearly independent** if the vector equation

  $$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \vec{\mathbf{0}}$$

  has only the **trivial solution** (of all zeros): $c_1 = 0, c_2 = 0, \cdots, c_k = 0$

  If there are also nontrivial solutions, then $S$ is called **linearly dependent**.

- **SPANNING SET & LINEAR INDEPENDENCE TESTS (SIMPLIFIED PROCEDURE):**

  TASK: Determine whether $S = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$ spans vector space $V$ and/or $S$ is linearly independent.

  (1) Form matrix $A$ with $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ as its columns: $A = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k \\ | & | & & | \end{bmatrix}$

  (2) Perform Gauss-Jordan Elimination on matrix $A$: $A \xrightarrow{Gauss-Jordan} \text{RREF}(A)$

  ($\star$) If **every row** of $\text{RREF}(A)$ **contains a pivot**, then $S$ spans $V$.

  ($\star$) If $\text{RREF}(A)$ **contains row(s) of all zeros**, then $S$ does not span $V$.

  ($\star$) If **every column** of $\text{RREF}(A)$ **contains a pivot**, $S$ is linearly independent.

  ($\star$) If $\text{RREF}(A)$ **contains column(s) without a pivot**, $S$ is linearly dependent.

  Non-pivot columns of $A$ are linear combinations of pivot columns of $A$.

  Such linear comb's are expressed in the non-pivot columns of $\text{RREF}(A)$.

  NOTE: For **polynomials**, form each column of matrix $A$ using the **coefficients** of each polynomial.

**EX 4.4.1:** Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $S = \left\{ \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix} \right\} \equiv \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \subseteq \mathbb{R}^3$.

(a) Write $\mathbf{u}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, if possible.

(b) Is $\mathbf{u} \in \text{span}(S)$?

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**EX 4.4.2:** Let $\mathbf{u} = \begin{bmatrix} 4 \\ 1 \\ 14 \end{bmatrix}$ and $S = \left\{ \begin{bmatrix} 3 \\ 1 \\ 11 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\} \equiv \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \subseteq \mathbb{R}^3$.

(a) Write $\mathbf{u}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, if possible.

(b) Is $\mathbf{u} \in \text{span}(S)$?

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**EX 4.4.3:** Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $S = \left\{ \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ -9 \\ 9 \end{bmatrix} \right\} \equiv \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \subseteq \mathbb{R}^3$.

(a) Write $\mathbf{u}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, if possible.

(b) Is $\mathbf{u} \in \text{span}(S)$?

**EX 4.4.4:** Let $r(t) = 3t + 2$ and $S = \{2t^2 - 3t + 6,\ 4t^2 + t,\ t^2 + 5t - 1\} \equiv \{p_1(t), p_2(t), p_3(t)\} \subseteq P_2$.

(a) Write $r(t)$ as a linear combination of $p_1(t), p_2(t), p_3(t)$, if possible.

(b) Is $r(t) \in \text{span}(S)$?

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**EX 4.4.5:** Let $r(t) = -8t^2 - 16t + 24$ and $S = \{t^2 + 2t - 3,\ -4t^2 - 8t + 12,\ 2t^2 + 4t - 6\} \equiv \{p_1(t), p_2(t), p_3(t)\} \subseteq P_2$.

(a) Write $r(t)$ as a linear combination of $p_1(t), p_2(t), p_3(t)$, if possible.

(b) Is $r(t) \in \text{span}(S)$?

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**EX 4.4.6:** Let $r(t) = 3t^2 + t + 5$ and $S = \{t^2 + 2t - 3,\ \ -4t^2 - 8t + 12,\ \ 2t^2 + 4t - 6\} \equiv \{p_1(t), p_2(t), p_3(t)\} \subseteq P_2$.

(a) Write $r(t)$ as a linear combination of $p_1(t), p_2(t), p_3(t)$, if possible.

(b) Is $r(t) \in \text{span}(S)$?

**EX 4.4.7:** Let $S = \left\{ \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix} \right\} \equiv \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \subseteq \mathbb{R}^3$.

(a) Does $S$ span $\mathbb{R}^3$?

(b) Is $S$ linearly independent or dependent?

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**EX 4.4.8:** Let $S = \left\{ \begin{bmatrix} 3 \\ 1 \\ 11 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\} \equiv \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \subseteq \mathbb{R}^3$.

(a) Does $S$ span $\mathbb{R}^3$?

(b) Is $S$ linearly independent or dependent?

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**EX 4.4.9:** Let $S = \left\{ \begin{bmatrix} r + 3t \\ 2s - 4t \end{bmatrix} : r, s, t \in \mathbb{R} \right\}$. Find vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ such that $S = \text{span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.

**EX 4.4.10:** Let $S = \{2t^2 - 3t + 6,\ 4t^2 + t,\ t^2 + 5t - 1\} \equiv \{p_1(t), p_2(t), p_3(t)\} \subseteq P_2$.

(a) Does $S$ span $P_2$?

(b) Is $S$ linearly independent or dependent?

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**EX 4.4.11:** Let $S = \{t^2 + 2t - 3,\quad -4t^2 - 8t + 12,\quad 2t^2 + 4t - 6\} \equiv \{p_1(t), p_2(t), p_3(t)\} \subseteq P_2$.

(a) Does $S$ span $P_2$?

(b) Is $S$ linearly independent or dependent?