EX 4.5.1: Let $\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix} \right\} \equiv \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \subseteq \mathbb{R}^3$.

Is $\mathcal{S}$ a basis for $\mathbb{R}^3$? (Justify answer) If not, find a basis $\mathcal{B}$ for the subspace spanned by $\mathcal{S}$ & find dim(span$\{\mathcal{S}\}$).

$$A = \begin{bmatrix} \boxed{1} & 3 & 3 \\ 3 & 2 & 3 \\ -3 & 3 & 2 \end{bmatrix} \xrightarrow[3R_1+R_3\to R_3]{(-3)R_1+R_2\to R_2} \begin{bmatrix} \boxed{1} & 3 & 3 \\ 0 & -7 & -6 \\ 0 & 12 & 11 \end{bmatrix} \xrightarrow[7R_3\to R_3]{12R_2\to R_2} \begin{bmatrix} \boxed{1} & 3 & 3 \\ 0 & -84 & -72 \\ 0 & 84 & 77 \end{bmatrix} \xrightarrow{R_2+R_3\to R_3} \begin{bmatrix} \boxed{1} & 3 & 3 \\ 0 & -84 & -72 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\xrightarrow[(\frac{1}{5})R_3\to R_3]{(-\frac{1}{12})R_2\to R_2} \begin{bmatrix} \boxed{1} & 3 & 3 \\ 0 & 7 & 6 \\ 0 & 0 & \boxed{1} \end{bmatrix} \xrightarrow[(-3)R_3+R_1\to R_1]{(-6)R_3+R_2\to R_2} \begin{bmatrix} \boxed{1} & 3 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix} \xrightarrow{(\frac{1}{7})R_2\to R_2} \begin{bmatrix} \boxed{1} & 3 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix} \xrightarrow{(-3)R_2+R_1\to R_1} \underbrace{\begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}}_{\text{RREF}(A)}$$

Since every row of RREF(A) has a pivot, $\mathcal{S}$ spans $\mathbb{R}^3$.

Since every column of RREF(A) has a pivot, $\mathcal{S}$ is linearly independent. $\therefore$ $\boxed{\mathcal{S} \text{ is a basis for } \mathbb{R}^3}$

EX 4.5.2: Let $\mathcal{S} = \left\{ \begin{bmatrix} 3 \\ 1 \\ 11 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\} \equiv \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \subseteq \mathbb{R}^3$.

Is $\mathcal{S}$ a basis for $\mathbb{R}^3$? (Justify answer) If not, find a basis $\mathcal{B}$ for the subspace spanned by $\mathcal{S}$ & find dim(span$\{\mathcal{S}\}$).

$$A = \begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} 3 & 2 & -1 \\ 1 & -2 & 2 \\ 11 & 2 & 1 \end{bmatrix} \xrightarrow{R_1\leftrightarrow R_2} \begin{bmatrix} \boxed{1} & -2 & 2 \\ 3 & 2 & -1 \\ 11 & 2 & 1 \end{bmatrix} \xrightarrow[(-11)R_1+R_3\to R_3]{(-3)R_1+R_2\to R_2} \begin{bmatrix} \boxed{1} & -2 & 2 \\ 0 & 8 & -7 \\ 0 & 24 & -21 \end{bmatrix}$$

$$\xrightarrow{(-3)R_2+R_3\to R_3} \begin{bmatrix} \boxed{1} & -2 & 2 \\ 0 & 8 & -7 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{(\frac{1}{8})R_2\to R_2} \begin{bmatrix} \boxed{1} & -2 & 2 \\ 0 & \boxed{1} & -7/8 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{2R_2+R_1\to R_1} \begin{bmatrix} \boxed{1} & 0 & 1/4 \\ 0 & \boxed{1} & -7/8 \\ 0 & 0 & 0 \end{bmatrix} = \text{RREF}(A)$$

Since at least one row of RREF(A) has no pivot, $\mathcal{S}$ does <u>not</u> span $\mathbb{R}^3$.

Since at least one column of RREF(A) has no pivot, $\mathcal{S}$ is <u>not</u> linearly independent. $\therefore$ $\boxed{\mathcal{S} \text{ is not a basis for } \mathbb{R}^3}$

$\therefore$ Basis $\mathcal{B} = \{\text{pivot columns of } A\} = \{\mathbf{v}_1, \mathbf{v}_2\} = \boxed{\left\{ \begin{bmatrix} 3 \\ 1 \\ 11 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix} \right\}} \implies \boxed{\dim(\text{span}\{\mathcal{S}\}) = 2}$

EX 4.5.3: Let $\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ -6 \end{bmatrix}, \begin{bmatrix} -2 \\ -6 \\ 4 \end{bmatrix} \right\} \equiv \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \subseteq \mathbb{R}^3$.

Is $\mathcal{S}$ a basis for $\mathbb{R}^3$? (Justify answer) If not, find a basis $\mathcal{B}$ for the subspace spanned by $\mathcal{S}$ & find dim(span$\{\mathcal{S}\}$).

$$A = \begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} \boxed{1} & 3 & -2 \\ 3 & 9 & -6 \\ -2 & -6 & 4 \end{bmatrix} \xrightarrow[2R_1+R_3\to R_3]{(-3)R_1+R_2\to R_2} \begin{bmatrix} \boxed{1} & 3 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{RREF}(A)$$

Since at least one row of RREF(A) has no pivot, $\mathcal{S}$ does <u>not</u> span $\mathbb{R}^3$.

Since at least one column of RREF(A) has no pivot, $\mathcal{S}$ is <u>not</u> linearly independent. $\therefore$ $\boxed{\mathcal{S} \text{ is not a basis for } \mathbb{R}^3}$

$\therefore$ Basis $\mathcal{B} = \{\text{pivot columns of } A\} = \{\mathbf{v}_1\} = \boxed{\left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} \right\}} \implies \boxed{\dim(\text{span}\{\mathcal{S}\}) = 1}$

EX 4.5.4: Let $\mathcal{S} = \{2t^2 - 3t + 6,\ 4t^2 + t,\ t^2 + 5t - 1\} \equiv \{p_1(t), p_2(t), p_3(t)\} \subseteq P_2$.

Is $\mathcal{S}$ a basis for P_2? (Justify answer) If not, find a basis $\mathcal{B}$ for the subspace spanned by $\mathcal{S}$ & find dim(span$\{\mathcal{S}\}$).

$$A = \begin{bmatrix} | & | & | \\ p_1(t) & p_2(t) & p_3(t) \\ | & | & | \end{bmatrix} = \begin{bmatrix} 2 & 4 & 1 \\ -3 & 1 & 5 \\ 6 & 0 & -1 \end{bmatrix} \xrightarrow[2R_2 \to R_2]{3R_1 \to R_1} \begin{bmatrix} 6 & 12 & 3 \\ -6 & 2 & 10 \\ 6 & 0 & -1 \end{bmatrix} \xrightarrow[(-1)R_1 + R_3 \to R_3]{R_1 + R_2 \to R_2} \begin{bmatrix} 6 & 12 & 3 \\ 0 & 14 & 13 \\ 0 & -12 & -4 \end{bmatrix}$$

$$\xrightarrow[(-\frac{1}{4})R_3 \to R_3]{(\frac{1}{3})R_1 \to R_1} \begin{bmatrix} 2 & 4 & 1 \\ 0 & 14 & 13 \\ 0 & 3 & 1 \end{bmatrix} \xrightarrow[14R_3 \to R_3]{3R_2 \to R_2} \begin{bmatrix} 2 & 4 & 1 \\ 0 & 42 & 39 \\ 0 & 42 & 14 \end{bmatrix} \xrightarrow{(-1)R_2 + R_3 \to R_3} \begin{bmatrix} 2 & 4 & 1 \\ 0 & 42 & 39 \\ 0 & 0 & -25 \end{bmatrix} \xrightarrow[(-\frac{1}{25})R_3 \to R_3]{(\frac{1}{3})R_2 \to R_2} \begin{bmatrix} 2 & 4 & 1 \\ 0 & 14 & 13 \\ 0 & 0 & \boxed{1} \end{bmatrix}$$

$$\xrightarrow[(-1)R_3 + R_1 \to R_1]{(-13)R_3 + R_2 \to R_2} \begin{bmatrix} 2 & 4 & 0 \\ 0 & 14 & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix} \xrightarrow[(\frac{1}{14})R_2 \to R_2]{(\frac{1}{2})R_1 \to R_1} \begin{bmatrix} \boxed{1} & 2 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix} \xrightarrow{(-2)R_2 + R_1 \to R_1} \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix} = \text{RREF}(A)$$

Since every row of RREF(A) has a pivot, $\mathcal{S}$ spans P_2.

Since every column of RREF(A) has a pivot, $\mathcal{S}$ is linearly independent. $\therefore$ $\boxed{\mathcal{S} \text{ is a basis for } P_2}$

EX 4.5.5: Let $\mathcal{S} = \{t^2 + 2t - 3,\ -4t^2 - 8t + 12,\ 2t^2 + 4t - 6\} \equiv \{p_1(t), p_2(t), p_3(t)\} \subseteq P_2$.

Is $\mathcal{S}$ a basis for P_2? (Justify answer) If not, find a basis $\mathcal{B}$ for the subspace spanned by $\mathcal{S}$ & find dim(span$\{\mathcal{S}\}$).

$$A = \begin{bmatrix} | & | & | \\ p_1(t) & p_2(t) & p_3(t) \\ | & | & | \end{bmatrix} = \begin{bmatrix} \boxed{1} & -4 & 2 \\ 2 & -8 & 4 \\ -3 & 12 & -6 \end{bmatrix} \xrightarrow[3R_1 + R_3 \to R_3]{(-2)R_1 + R_2 \to R_2} \begin{bmatrix} \boxed{1} & -4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{RREF}(A)$$

Since at least one row of RREF(A) has no pivot, $\mathcal{S}$ does not span P_2.

Since at least one column of RREF(A) has no pivot, $\mathcal{S}$ is not linearly independent. $\therefore$ $\boxed{\mathcal{S} \text{ is not a basis for } P_2}$

$\therefore$ Basis $\mathcal{B} = \{\text{pivot columns of } A\} = \{p_1(t)\} = \boxed{\{t^2 + 2t - 3\}} \implies \boxed{\dim(\text{span}\{\mathcal{S}\}) = 1}$

NOTE: $p_2(t)$ is not part of the basis $\mathcal{B}$ since it's a linear combination of $p_1(t)$: $p_2(t) = (-4)p_1(t)$

NOTE: $p_3(t)$ is not part of the basis $\mathcal{B}$ since it's a linear combination of $p_1(t)$: $p_3(t) = 2p_1(t)$

EX 4.5.6: Let $W = \left\{ \begin{bmatrix} r + 3t \\ 2s - 4t \end{bmatrix} : r, s, t \in \mathbb{R} \right\}$ be a subspace of $\mathbb{R}^2$. Find a basis $\mathcal{B}$ for W & find $\dim(W)$.

$$\begin{bmatrix} r + 3t \\ 2s - 4t \end{bmatrix} = \begin{bmatrix} r \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2s \end{bmatrix} + \begin{bmatrix} 3t \\ -4t \end{bmatrix} \quad \text{(Undo Vector Addition)}$$

$$= r\begin{bmatrix} 1 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 2 \end{bmatrix} + t\begin{bmatrix} 3 \\ -4 \end{bmatrix} \quad \text{(Undo Scalar Multiplication)}$$

$$\therefore\ W = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \end{bmatrix} \right\} \equiv \text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$$

$$A = \begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{bmatrix} = \begin{bmatrix} \boxed{1} & 0 & 3 \\ 0 & 2 & -4 \end{bmatrix} \xrightarrow{(\frac{1}{2})R_2 \to R_2} \begin{bmatrix} \boxed{1} & 0 & 3 \\ 0 & \boxed{1} & -2 \end{bmatrix} = \text{RREF}(A)$$

$\therefore$ Basis $\mathcal{B} = \{\text{pivot columns of } A\} = \{\mathbf{v}_1, \mathbf{v}_2\} = \boxed{\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix} \right\}} \implies \boxed{\dim(W) = 2}$

NOTE: $\mathbf{v}_3$ is not part of the basis $\mathcal{B}$ since it's a linear combination of $\mathbf{v}_1$ & $\mathbf{v}_2$: $\mathbf{v}_3 = 3\mathbf{v}_1 + (-2)\mathbf{v}_2$