VECTOR SPACES: BASIS, DIMENSION [LARSON 4.5]

- **BASIS FOR A VECTOR SPACE (DEFINITION):**

  Let V be a vector space and $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \subseteq V$

  Then $\mathcal{S}$ is a **basis** for V if: $\mathcal{S}$ spans V **AND** $\mathcal{S}$ is linearly independent

  Moreover, each vector in a basis is called a **basis vector**.

- **DIMENSION OF A VECTOR SPACE (DEFINITION):**

  Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a basis for vector space V.

  Then the **dimension** of V is the # of basis vectors in $\mathcal{B}$: $\dim(V) = k$

  Let vector space Z contain only the **zero vector**. Then $\dim(Z) := 0$

- **FINITE & INFINITE DIMENSIONAL VECTOR SPACES:**

  Let $\mathcal{B}$ be a basis for vector space V. Then:

  V is **finite dimensional** if basis $\mathcal{B}$ contains a **finite** # of basis vectors.

  V is **infinite dimensional** if basis $\mathcal{B}$ contains an **infinite** # of basis vectors.

  Vector spaces $\mathbb{R}$, $\mathbb{R}^n$, $\mathbb{R}^{m\times n}$, P_n are each finite dimensional.

  Vector spaces $C[a,b]$, $C^1[a,b]$, $C^2[a,b]$, $C^k[a,b]$, P are each infinite dimensional.

  Infinite dimensional vector spaces are beyond the scope of this chapter.

- **STANDARD BASES:** Many vector spaces have a "intuitive" basis, called the **standard basis**.

  NOTATION: A standard basis is denoted by $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_k\}$, where $\mathbf{e}_j$ is the j^{th} standard basis vector.

  With a standard basis, the coefficients in the linear combination are simply the entries of the vector:

  $$\star \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\mathbf{e}_1} + x_2 \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\mathbf{e}_2} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$$

  $$\star \quad a_0 + a_1 t + a_2 t^2 + a_3 t^3 = a_0 \underbrace{(1)}_{\mathbf{e}_1} + a_1 \underbrace{(t)}_{\mathbf{e}_2} + a_2 \underbrace{(t^2)}_{\mathbf{e}_3} + a_3 \underbrace{(t^3)}_{\mathbf{e}_4} = a_0\mathbf{e}_1 + a_1\mathbf{e}_2 + a_2\mathbf{e}_3 + a_3\mathbf{e}_4$$

- **FINDING A BASIS FOR A SUBSPACE SPANNED BY A SET:**

  TASK: Find a basis $\mathcal{B}$ for the subspace spanned by $\mathcal{S} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$.

  (1) Form matrix A with $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ as its columns: $A = \begin{bmatrix} | & | & & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k \\ | & | & & | \end{bmatrix}$

  (2) Perform Gauss-Jordan Elimination on matrix A: $A \xrightarrow{Gauss-Jordan} \text{RREF}(A)$

  (3) Basis $\mathcal{B} = \{\text{pivot columns of } A\}$.

  NOTE: For **polynomials**, form each column of matrix A using the **coefficients** of each polynomial.

- **FINDING A BASIS FOR A SUBSPACE WRITTEN IN TERMS OF PARAMETERS:**

  TASK: Find a basis $\mathcal{B}$ for the subspace $W = \{\mathbf{w}_1 : s, t, \ldots \in \mathbb{R}\}$ of vector space V.

  (1) "Undo" any vector addition by writing $\mathbf{w}_1$ as a sum of vectors, each of which has its own parameter.

  (2) "Undo" any scalar multiplication by factoring out each parameter from each vector.

  (3) The resulting set of vectors span W.

  (4) Apply the previous procedure to this set of vectors to remove any linearly dependent vectors.

EX 4.5.1: Let $\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix} \right\} \equiv \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \subseteq \mathbb{R}^3$.

Is $\mathcal{S}$ a basis for $\mathbb{R}^3$? (Justify answer) If not, find a basis $\mathcal{B}$ for the subspace spanned by $\mathcal{S}$ & find $\dim(\text{span}\{\mathcal{S}\})$.

EX 4.5.2: Let $\mathcal{S} = \left\{ \begin{bmatrix} 3 \\ 1 \\ 11 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\} \equiv \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \subseteq \mathbb{R}^3$.

Is $\mathcal{S}$ a basis for $\mathbb{R}^3$? (Justify answer) If not, find a basis $\mathcal{B}$ for the subspace spanned by $\mathcal{S}$ & find $\dim(\text{span}\{\mathcal{S}\})$.

EX 4.5.3: Let $\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 9 \\ -6 \end{bmatrix}, \begin{bmatrix} -2 \\ -6 \\ 4 \end{bmatrix} \right\} \equiv \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} \subseteq \mathbb{R}^3$.

Is $\mathcal{S}$ a basis for $\mathbb{R}^3$? (Justify answer) If not, find a basis $\mathcal{B}$ for the subspace spanned by $\mathcal{S}$ & find $\dim(\text{span}\{\mathcal{S}\})$.

EX 4.5.4: Let $\mathcal{S} = \{2t^2 - 3t + 6,\ 4t^2 + t,\ t^2 + 5t - 1\} \equiv \{p_1(t), p_2(t), p_3(t)\} \subseteq P_2$.

Is $\mathcal{S}$ a basis for P_2? (Justify answer) If not, find a basis $\mathcal{B}$ for the subspace spanned by $\mathcal{S}$ & find dim(span$\{\mathcal{S}\}$).

EX 4.5.5: Let $\mathcal{S} = \{t^2 + 2t - 3,\ -4t^2 - 8t + 12,\ 2t^2 + 4t - 6\} \equiv \{p_1(t), p_2(t), p_3(t)\} \subseteq P_2$.

Is $\mathcal{S}$ a basis for P_2? (Justify answer) If not, find a basis $\mathcal{B}$ for the subspace spanned by $\mathcal{S}$ & find dim(span$\{\mathcal{S}\}$).

EX 4.5.6: Let $W = \left\{ \begin{bmatrix} r + 3t \\ 2s - 4t \end{bmatrix} : r, s, t \in \mathbb{R} \right\}$ be a subspace of $\mathbb{R}^2$. Find a basis $\mathcal{B}$ for W & find $\dim(W)$.