EX 4.6.1: Let $A = \begin{bmatrix} 1 & 3 & 3 \\ 3 & 2 & 3 \\ -3 & 3 & 2 \end{bmatrix}$.

(a) Find bases & the dimensions for RowSp(A) and ColSp(A).
(b) Find RowSp(A) and ColSp(A).
(c) Find rank(A). Does A have full row rank? Full column rank?

$$A = \begin{bmatrix} \boxed{1} & 3 & 3 \\ 3 & 2 & 3 \\ -3 & 3 & 2 \end{bmatrix} \xrightarrow[3R_1+R_3\to R_3]{(-3)R_1+R_2\to R_2} \begin{bmatrix} \boxed{1} & 3 & 3 \\ 0 & -7 & -6 \\ 0 & 12 & 11 \end{bmatrix} \xrightarrow[7R_3\to R_3]{12R_2\to R_2} \begin{bmatrix} \boxed{1} & 3 & 3 \\ 0 & -84 & -72 \\ 0 & 84 & 77 \end{bmatrix} \xrightarrow{R_2+R_3\to R_3} \begin{bmatrix} \boxed{1} & 3 & 3 \\ 0 & -84 & -72 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\xrightarrow[(\frac{1}{5})R_3\to R_3]{(-\frac{1}{12})R_2\to R_2} \begin{bmatrix} \boxed{1} & 3 & 3 \\ 0 & 7 & 6 \\ 0 & 0 & \boxed{1} \end{bmatrix} \xrightarrow[(-3)R_3+R_1\to R_1]{(-6)R_3+R_2\to R_2} \begin{bmatrix} \boxed{1} & 3 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix} \xrightarrow{(\frac{1}{7})R_2\to R_2} \begin{bmatrix} \boxed{1} & 3 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix} \xrightarrow{(-3)R_2+R_1\to R_1} \underbrace{\begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ 0 & 0 & \boxed{1} \end{bmatrix}}_{\text{RREF}(A)}$$

(a) Basis for RowSp(A) = {pivot rows of RREF(A)} = $\boxed{\{(1,0,0),(0,1,0),(0,0,1)\}}$ $\implies$ dim RowSp(A) = $\boxed{3}$

(a) Basis for ColSp(A) = {pivot columns of (A)} = $\boxed{\left\{(1,3,-3)^T,(3,2,3)^T,(3,3,2)^T\right\}}$ $\implies$ dim ColSp(A) = $\boxed{3}$

(b) $\boxed{\text{RowSp}(A) = \text{span}\{(1,0,0),(0,1,0),(0,0,1)\}}$, $\boxed{\text{ColSp}(A) = \text{span}\left\{\begin{bmatrix} 1 \\ 3 \\ -3 \end{bmatrix}, \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix}\right\}}$

(c) rank(A) = [# pivots in RREF(A)] = $\boxed{3}$ $\boxed{\text{Yes \& Yes, since every row \& column of RREF}(A)\text{ has a pivot}}$

EX 4.6.2: Let $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & -2 & 2 \\ 11 & 2 & 1 \end{bmatrix}$.

(a) Find bases & the dimensions for RowSp(A) and ColSp(A).
(b) Find RowSp(A) and ColSp(A).
(c) Find rank(A). Does A have full row rank? Full column rank?

$$A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & -2 & 2 \\ 11 & 2 & 1 \end{bmatrix} \xrightarrow{R_1\leftrightarrow R_2} \begin{bmatrix} \boxed{1} & -2 & 2 \\ 3 & 2 & -1 \\ 11 & 2 & 1 \end{bmatrix} \xrightarrow[(-11)R_1+R_3\to R_3]{(-3)R_1+R_2\to R_2} \begin{bmatrix} \boxed{1} & -2 & 2 \\ 0 & 8 & -7 \\ 0 & 24 & -21 \end{bmatrix} \xrightarrow{(-3)R_2+R_3\to R_3} \begin{bmatrix} \boxed{1} & -2 & 2 \\ 0 & 8 & -7 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{(\frac{1}{8})R_2\to R_2} \begin{bmatrix} \boxed{1} & -2 & 2 \\ 0 & \boxed{1} & -7/8 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{2R_2+R_1\to R_1} \begin{bmatrix} \boxed{1} & 0 & 1/4 \\ 0 & \boxed{1} & -7/8 \\ 0 & 0 & 0 \end{bmatrix} = \text{RREF}(A)$$

(a) Basis for RowSp(A) = {pivot rows of RREF(A)} = $\boxed{\{(1,0,1/4),(0,1,-7/8)\}}$ $\implies$ dim RowSp(A) = $\boxed{2}$

(a) Basis for ColSp(A) = {pivot columns of (A)} = $\boxed{\left\{(3,1,11)^T,(2,-2,2)^T\right\}}$ $\implies$ dim ColSp(A) = $\boxed{2}$

(b) $\boxed{\text{RowSp}(A) = \text{span}\{(1,0,1/4),(0,1,-7/8)\}}$, $\boxed{\text{ColSp}(A) = \text{span}\left\{(3,1,11)^T,(2,-2,2)^T\right\}}$

(c) rank(A) = [# pivots in RREF(A)] = $\boxed{2}$ $\boxed{\text{No \& No, since at least one row \& column of RREF}(A)\text{ has no pivot}}$

EX 4.6.3: Let $A = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 9 & -6 \\ -2 & -6 & 4 \end{bmatrix}$.

(a) Is $\mathbf{b}_1 = (4,12,-8)^T \in$ ColSp(A)? Is linear system $A\mathbf{x} = \mathbf{b}_1$ consistent?
(b) Is $\mathbf{b}_2 = (1,2,3)^T \in$ ColSp(A)? Is linear system $A\mathbf{x} = \mathbf{b}_2$ consistent?

(a) $$[A \mid \mathbf{b}_1] = \left[\begin{array}{ccc|c} \boxed{1} & 3 & -2 & 4 \\ 3 & 9 & -6 & 12 \\ -2 & -6 & 4 & -8 \end{array}\right] \xrightarrow[2R_1+R_3\to R_3]{(-3)R_1+R_2\to R_2} \left[\begin{array}{ccc|c} \boxed{1} & 3 & -2 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = [\text{RREF}(A) \mid \widetilde{\mathbf{b}}_1]$$

Since a CONTRADICTION never occurred, $\boxed{\text{Yes, } \mathbf{b}_1 \in \text{ColSp}(A)}$ $\implies$ $\boxed{\text{Yes, } A\mathbf{x} = \mathbf{b}_1 \text{ \underline{is} consistent}}$

(b) $$[A \mid \mathbf{b}_2] = \left[\begin{array}{ccc|c} \boxed{1} & 3 & -2 & 1 \\ 3 & 9 & -6 & 2 \\ -2 & -6 & 4 & 3 \end{array}\right] \xrightarrow[2R_1+R_3\to R_3]{(-3)R_1+R_2\to R_2} \left[\begin{array}{ccc|c} \boxed{1} & 3 & -2 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 5 \end{array}\right] \begin{array}{l} \implies 0 = -1 \quad \leftarrow \text{CONTRADICTION!} \\ \implies 0 = \ \ 5 \quad \leftarrow \text{CONTRADICTION!} \end{array}$$

Since at least one CONTRADICTION occurred, $\boxed{\text{No, } \mathbf{b}_2 \notin \text{ColSp}(A)}$ $\implies$ $\boxed{\text{No, } A\mathbf{x} = \mathbf{b}_2 \text{ is \underline{not} consistent (i.e. no soln)}}$