EX 4.6.4: Let $A = \begin{bmatrix} 1 & 3 & 3 \\ 3 & 2 & 3 \\ -3 & 3 & 2 \end{bmatrix}$. (a) Find a basis for NulSp(A). (b) Find NulSp(A). (c) Find rank(A) and nullity(A).

(To conserve space, the Gauss-Jordan elimination steps are omitted – see solution to EX 4.6.1 to see the omitted steps.)

$$[A \mid \vec{\mathbf{0}}] = \left[\begin{array}{ccc|c} \boxed{1} & 3 & 3 & 0 \\ 3 & 2 & 3 & 0 \\ -3 & 3 & 2 & 0 \end{array}\right] \xrightarrow{Gauss-Jordan} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 \end{array}\right] = [\text{RREF}(A) \mid \vec{\mathbf{0}}]$$

(a) Since every column of RREF(A) has a pivot, a basis for NulSp(A) = $\boxed{\{\vec{\mathbf{0}}\}}$ (b) NulSp(A) = span$\{\vec{\mathbf{0}}\}$ = $\boxed{\{\vec{\mathbf{0}}\}}$

REMARK: span$\{\vec{\mathbf{0}}\}$ = {All linear combinations of $\vec{\mathbf{0}}$} = {All scalar multiples of $\vec{\mathbf{0}}$} = $\{\vec{\mathbf{0}}\}$

(c) rank(A) = [# pivot columns in RREF(A)] = $\boxed{3}$ nullity(A) = [# non-pivot columns of RREF(A)] = $\boxed{0}$

EX 4.6.5: Let $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & -2 & 2 \\ 11 & 2 & 1 \end{bmatrix}$. (a) Find a basis for NulSp(A). (b) Find NulSp(A). (c) Find rank(A) and nullity(A).

$$[A \mid \vec{\mathbf{0}}] = \left[\begin{array}{ccc|c} 3 & 2 & -1 & 0 \\ 1 & -2 & 2 & 0 \\ 11 & 2 & 1 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} \boxed{1} & -2 & 2 & 0 \\ 3 & 2 & -1 & 0 \\ 11 & 2 & 1 & 0 \end{array}\right] \xrightarrow[(-11)R_1+R_3 \to R_3]{(-3)R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} \boxed{1} & -2 & 2 & 0 \\ 0 & 8 & -7 & 0 \\ 0 & 24 & -21 & 0 \end{array}\right]$$

$$\xrightarrow{(-3)R_2+R_3 \to R_3} \left[\begin{array}{ccc|c} \boxed{1} & -2 & 2 & 0 \\ 0 & 8 & -7 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{(\frac{1}{8})R_2 \to R_2} \left[\begin{array}{ccc|c} \boxed{1} & -2 & 2 & 0 \\ 0 & \boxed{1} & -7/8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{2R_2+R_1 \to R_1} \underbrace{\left[\begin{array}{ccc|c} \boxed{1} & 0 & 1/4 & 0 \\ 0 & \boxed{1} & -7/8 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]}_{[\text{RREF}(A) \mid \vec{\mathbf{0}}]}$$

There's no pivot in 3rd Column of RREF(A), so pick convenient parameterization for 3rd unknown: Let $x_3 = 8t$

$$[\text{RREF}(A) \mid \vec{\mathbf{0}}] \iff \begin{cases} x_1 \quad + \frac{1}{4}x_3 = 0 \\ x_2 - \frac{7}{8}x_3 = 0 \end{cases} \implies \begin{array}{c} x_1 = -\frac{1}{4}(8t) = -2t \\ x_2 = \frac{7}{8}(8t) = 7t \\ x_3 = 8t \end{array} \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -2 \\ 7 \\ 8 \end{bmatrix}$$

(a) A basis for NulSp(A) = $\boxed{\left\{ \begin{bmatrix} -2 \\ 7 \\ 8 \end{bmatrix} \right\}}$ (b) $\boxed{\text{NulSp}(A) = \text{span} \left\{ \begin{bmatrix} -2 \\ 7 \\ 8 \end{bmatrix} \right\}}$

(c) rank(A) = [# pivot columns in RREF(A)] = $\boxed{2}$ nullity(A) = [# non-pivot columns of RREF(A)] = $\boxed{1}$

EX 4.6.6: Let $A = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 9 & -6 \\ -2 & -6 & 4 \end{bmatrix}$. (a) Find a basis for NulSp(A). (b) Find NulSp(A). (c) Find rank(A) and nullity(A).

$$[A \mid \vec{\mathbf{0}}] = \left[\begin{array}{ccc|c} \boxed{1} & 3 & -2 & 0 \\ 3 & 9 & -6 & 0 \\ -2 & -6 & 4 & 0 \end{array}\right] \xrightarrow[2R_1+R_3 \to R_3]{(-3)R_1+R_2 \to R_2} \left[\begin{array}{ccc|c} \boxed{1} & 3 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = [\text{RREF}(A) \mid \vec{\mathbf{0}}]$$

Columns 2 & 3 of RREF(A) have no pivots, so pick parameterizations for 2nd & 3rd unknowns: Let $x_2 = s$, $x_3 = t$

$$[\text{RREF}(A) \mid \vec{\mathbf{0}}] \iff \{ x_1 + 3x_2 - 2x_3 = 0 \implies \begin{array}{c} x_1 = -3s + 2t \\ x_2 = s \\ x_3 = t \end{array} \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

(a) A basis for NulSp(A) = $\boxed{\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}}$ (b) $\boxed{\text{NulSp}(A) = \text{span} \left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}}$

(c) rank(A) = [# pivot columns in RREF(A)] = $\boxed{1}$ nullity(A) = [# non-pivot columns of RREF(A)] = $\boxed{2}$

EX 4.6.7: Find the homogeneous ($\mathbf{x}_h$) & particular ($\mathbf{x}_p$) solutions to the linear system $\begin{cases} x_1 + 3x_2 + 3x_3 = 1 \\ 3x_1 + 2x_2 + 3x_3 = 2 \\ -3x_1 + 3x_2 + 2x_3 = 3 \end{cases}$

(To conserve space, the Gauss-Jordan elimination steps are omitted – see solution to EX 4.6.1 to see the omitted steps.)

$$[A \mid \mathbf{b}] = \left[\begin{array}{ccc|c} \boxed{1} & 3 & 3 & 1 \\ 3 & 2 & 3 & 2 \\ -3 & 3 & 2 & 3 \end{array}\right] \xrightarrow{Gauss-Jordan} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & -2 \\ 0 & \boxed{1} & 0 & -5 \\ 0 & 0 & \boxed{1} & 6 \end{array}\right] = [\text{RREF}(A) \mid \widetilde{\mathbf{b}}]$$

$$[\text{RREF}(A) \mid \widetilde{\mathbf{b}}] \iff \begin{cases} x_1 = -2 \\ x_2 = -5 \\ x_3 = 6 \end{cases} \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ -5 \\ 6 \end{bmatrix} = \underbrace{\begin{bmatrix} -2 \\ -5 \\ 6 \end{bmatrix}}_{\mathbf{x}_p} + t \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{x}_h}, \text{ where } t \in \mathbb{R}$$

EX 4.6.8: Find the homogeneous ($\mathbf{x}_h$) & particular ($\mathbf{x}_p$) solutions to the linear system $\begin{cases} 3x_1 + 2x_2 - x_3 = 4 \\ x_1 - 2x_2 + 2x_3 = 1 \\ 11x_1 + 2x_2 + x_3 = 14 \end{cases}$

(To conserve space, the Gauss-Jordan elimination steps are omitted – see solution to EX 4.6.5 to see the omitted steps.)

$$[A \mid \mathbf{b}] = \left[\begin{array}{ccc|c} 3 & 2 & -1 & 4 \\ 1 & -2 & 2 & 1 \\ 11 & 2 & 1 & 14 \end{array}\right] \xrightarrow{Gauss-Jordan} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 1/4 & 5/4 \\ 0 & \boxed{1} & -7/8 & 1/8 \\ 0 & 0 & 0 & 0 \end{array}\right] = [\text{RREF}(A) \mid \widetilde{\mathbf{b}}]$$

There's no pivot in 3^{rd} Column of RREF(A), so pick convenient parameterization for 3^{rd} unknown: Let $x_3 = 8t$

$$[\text{RREF}(A) \mid \widetilde{\mathbf{b}}] \iff \begin{cases} x_1 + \frac{1}{4}(8t) = \frac{5}{4} \\ x_2 - \frac{7}{8}(8t) = \frac{1}{8} \\ x_3 = 8t \end{cases} \implies \begin{cases} x_1 = \frac{5}{4} - 2t \\ x_2 = \frac{1}{8} + 7t \\ x_3 = 8t \end{cases} \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 5/4 \\ 1/8 \\ 0 \end{bmatrix}}_{\mathbf{x}_p} + t \underbrace{\begin{bmatrix} -2 \\ 7 \\ 8 \end{bmatrix}}_{\mathbf{x}_h}$$

EX 4.6.9: Find the homogeneous ($\mathbf{x}_h$) & particular ($\mathbf{x}_p$) solutions to the linear system $\begin{cases} x_1 + 3x_2 - 2x_3 = 2 \\ 3x_1 + 9x_2 - 6x_3 = 6 \\ -2x_1 - 6x_2 + 4x_3 = -4 \end{cases}$

$$[A \mid \mathbf{b}] = \left[\begin{array}{ccc|c} \boxed{1} & 3 & -2 & 2 \\ 3 & 9 & -6 & 6 \\ -2 & -6 & 4 & -4 \end{array}\right] \xrightarrow{Gauss-Jordan} \left[\begin{array}{ccc|c} \boxed{1} & 3 & -2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = [\text{RREF}(A) \mid \widetilde{\mathbf{b}}]$$

Columns 2 & 3 of RREF(A) have no pivots, so pick parameterizations for 2^{nd} & 3^{rd} unknowns: Let $x_2 = s$, $x_3 = t$

$$[\text{RREF}(A) \mid \widetilde{\mathbf{b}}] \iff x_1 + 3x_2 - 2x_3 = 2 \implies \begin{cases} x_1 = 2 - 3s + 2t \\ x_2 = s \\ x_3 = t \end{cases} \implies \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{x}_p} + s \underbrace{\begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}}_{\mathbf{x}_h} + t \underbrace{\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}}_{\mathbf{x}_h}$$

SANITY CHECK: $A\mathbf{x}_h = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 9 & -6 \\ -2 & -6 & 4 \end{bmatrix} \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \vec{\mathbf{0}} \implies A\mathbf{x}_h = \vec{\mathbf{0}}$ as expected

SANITY CHECK: $A\mathbf{x}_h = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 9 & -6 \\ -2 & -6 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \vec{\mathbf{0}} \implies A\mathbf{x}_h = \vec{\mathbf{0}}$ as expected

SANITY CHECK: $A\mathbf{x}_p = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 9 & -6 \\ -2 & -6 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ -4 \end{bmatrix} = \mathbf{b} \implies A\mathbf{x}_p = \mathbf{b}$ as expected