ROW SPACE, COLUMN SPACE, NULL SPACE, RANK [LARSON 4.6]

- **ROW SPACE OF A MATRIX:** The **row space** of $m \times n$ matrix A is the subspace spanned by its rows:

$$\text{RowSp}(A) := \text{ span}\{\text{rows of } A\} \subseteq \mathbb{R}^n$$

- **COLUMN SPACE OF A MATRIX:** The **column space** of $m \times n$ matrix A is the subspace spanned by its columns:

$$\text{ColSp}(A) := \text{ span}\{\text{columns of } A\} \subseteq \mathbb{R}^m$$

- **FINDING BASES FOR THE ROW SPACE & COLUMN SPACE:**

  TASK: Find bases for RowSp(A) & ColSp(A) where matrix $A \in \mathbb{R}^{m \times n}$.

  (1) Perform Gauss-Jordan Elimination on matrix A.

  ($\star$) The **pivot rows** of **RREF**(A) form a basis for RowSp(A).

  ($\star$) The **pivot columns** of A form a basis for ColSp(A).

- **RANK OF A MATRIX:** Let $A \in \mathbb{R}^{m \times n}$.

  Then the **rank** of A is the dimension of the column (row) space of A:

$$\text{rank}(A) := \dim \text{ColSp}(A) = \dim \text{RowSp}(A)$$

  In other words, the **rank** of A is simply the # of **pivots** in **RREF**(A).

  Matrix A has **full row rank** $\iff$ $\text{rank}(A) = m$

  Matrix A has **full column rank** $\iff$ $\text{rank}(A) = n$

- **COLUMN SPACE & CONSISTENCY OF CORRESPONDING LINEAR SYSTEM:**

  Let $A \in \mathbb{R}^{m \times n}$ and $\mathbf{b} \in \mathbb{R}^n$.

  Then $m \times n$ linear system $A\mathbf{x} = \mathbf{b}$ is consistent $\iff$ $\mathbf{b} \in \text{ColSp}(A)$

- **NULL SPACE OF A MATRIX:** The **null space** of $m \times n$ matrix A is the set of all solutions to $A\mathbf{x} = \vec{\mathbf{0}}$:

$$\text{NulSp}(A) := \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \vec{\mathbf{0}}\}$$

  Moreover, NulSp(A) is a subspace of $\mathbb{R}^n$.

  The **nullity** of A is the dimension of its null space: $\text{nullity}(A) := \dim \text{NulSp}(A)$

- **FINDING A BASIS FOR THE NULL SPACE:**

  TASK: Find the nullspace & nullity of matrix $A \in \mathbb{R}^{m \times n}$.

  (1) Perform Gauss-Jordan Elimination on augemented matrix $[A \mid \vec{\mathbf{0}}]$

  (2) Assign unique parameters to the free variables.

  (3) Form resulting solution $\mathbf{x}$ to $A\mathbf{x} = \vec{\mathbf{0}}$ by interpreting rows of $[\text{RREF}(A) \mid \vec{\mathbf{0}}]$

  (4) "Undo" vector addition by placing each parameter into its own vector.

  (5) "Undo" scalar multiplication by factoring the parameter from each vector.

  ($\star$) The resulting vectors form a basis for the nullspace of A.

  ($\star$) nullity(A) = # basis vectors for the nullspace of A.

  ($\star$) If the only solution to $A\mathbf{x} = \vec{\mathbf{0}}$ is $\vec{\mathbf{0}}$, then $\text{NulSp}(A) = \{\vec{\mathbf{0}}\}$ & $\text{nullity}(A) = 0$.

- **NULLSPACE & SOLVING $A\mathbf{x} = \mathbf{b}$:** Let $A \in \mathbb{R}^{m \times n}$ where $m \leq n$ (i.e. A is square or "short & wide" rectangular)

  Let **non-homogeneous** linear system $A\mathbf{x} = \mathbf{b}$ have **infinitely many** solutions.

  Then, the solution to $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$ where

  $\mathbf{x}_p$ is the **particular solution** to $A\mathbf{x} = \mathbf{b}$

  $\mathbf{x}_h$ is the **homogeneous solution** to $A\mathbf{x} = \vec{\mathbf{0}}$

EX 4.6.1: Let $A = \begin{bmatrix} 1 & 3 & 3 \\ 3 & 2 & 3 \\ -3 & 3 & 2 \end{bmatrix}$.

(a) Find bases & the dimensions for RowSp(A) and ColSp(A).
(b) Find RowSp(A) and ColSp(A).
(c) Find rank(A). Does A have full row rank? Full column rank?

EX 4.6.2: Let $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & -2 & 2 \\ 11 & 2 & 1 \end{bmatrix}$.

(a) Find bases & the dimensions for RowSp(A) and ColSp(A).
(b) Find RowSp(A) and ColSp(A).
(c) Find rank(A). Does A have full row rank? Full column rank?

EX 4.6.3: Let $A = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 9 & -6 \\ -2 & -6 & 4 \end{bmatrix}$.

(a) Is $\mathbf{b}_1 = (4, 12, -8)^T \in$ ColSp(A)? Is linear system $A\mathbf{x} = \mathbf{b}_1$ consistent?
(b) Is $\mathbf{b}_2 = (1, 2, 3)^T \in$ ColSp(A)? Is linear system $A\mathbf{x} = \mathbf{b}_2$ consistent?

EX 4.6.4: Let $A = \begin{bmatrix} 1 & 3 & 3 \\ 3 & 2 & 3 \\ -3 & 3 & 2 \end{bmatrix}$.

(a) Find a basis for NulSp(A).
(b) Find NulSp(A).
(c) Find rank(A) and nullity(A).

EX 4.6.5: Let $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & -2 & 2 \\ 11 & 2 & 1 \end{bmatrix}$.

(a) Find a basis for NulSp(A).
(b) Find NulSp(A).
(c) Find rank(A) and nullity(A).

EX 4.6.6: Let $A = \begin{bmatrix} 1 & 3 & -2 \\ 3 & 9 & -6 \\ -2 & -6 & 4 \end{bmatrix}$.

(a) Find a basis for NulSp(A).
(b) Find NulSp(A).
(c) Find rank(A) and nullity(A).

EX 4.6.7: Find the homogeneous ($\mathbf{x}_h$) & particular ($\mathbf{x}_p$) solutions to the linear system $\begin{cases} x_1 + 3x_2 + 3x_3 = 1 \\ 3x_1 + 2x_2 + 3x_3 = 2 \\ -3x_1 + 3x_2 + 2x_3 = 3 \end{cases}$

EX 4.6.8: Find the homogeneous ($\mathbf{x}_h$) & particular ($\mathbf{x}_p$) solutions to the linear system $\begin{cases} 3x_1 + 2x_2 - x_3 = 4 \\ x_1 - 2x_2 + 2x_3 = 1 \\ 11x_1 + 2x_2 + x_3 = 14 \end{cases}$

EX 4.6.9: Find the homogeneous ($\mathbf{x}_h$) & particular ($\mathbf{x}_p$) solutions to the linear system $\begin{cases} x_1 + 3x_2 - 2x_3 = 2 \\ 3x_1 + 9x_2 - 6x_3 = 6 \\ -2x_1 - 6x_2 + 4x_3 = -4 \end{cases}$