EX 4.7.1: Let basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ -9 \\ 12 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} \right\} \equiv \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Find $[\mathbf{x}]_{\mathcal{E}}$ if $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}$.

$$[\mathbf{x}]_{\mathcal{B}} = (4)\mathbf{v}_1 + (1)\mathbf{v}_2 + (-2)\mathbf{v}_3 = (4)\begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} + (1)\begin{bmatrix} -4 \\ -9 \\ 12 \end{bmatrix} + (-2)\begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ -14 \end{bmatrix} \quad \therefore \quad \boxed{[\mathbf{x}]_{\mathcal{E}} = \begin{bmatrix} 4 \\ -3 \\ -14 \end{bmatrix}}$$

NOTE: Here, $[\mathbf{x}]_{\mathcal{E}}$ can be written as $4\widehat{\mathbf{i}} - 3\widehat{\mathbf{j}} - 14\widehat{\mathbf{k}}$ as seen in Calculus III and Physics, but writing vectors in terms of $\widehat{\mathbf{i}}, \widehat{\mathbf{j}}, \widehat{\mathbf{k}}$ is not useful for this course, and hence is strongly discouraged.

EX 4.7.2: Let basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ -9 \\ 12 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} \right\} \equiv \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Find $[\mathbf{x}]_{\mathcal{B}}$ if $[\mathbf{x}]_{\mathcal{E}} = \begin{bmatrix} 1 \\ -8 \\ -5 \end{bmatrix}$.

$$\left[\, \mathcal{B} \mid [\mathbf{x}]_{\mathcal{E}} \,\right] = \left[\begin{array}{ccc|c} \boxed{1} & -4 & -2 & 1 \\ 2 & -9 & 1 & -8 \\ -3 & 12 & 7 & -5 \end{array}\right] \xrightarrow[3R_1+R_3\to R_3]{(-2)R_1+R_2\to R_2} \left[\begin{array}{ccc|c} \boxed{1} & -4 & -2 & 1 \\ 0 & -1 & 5 & -10 \\ 0 & 0 & \boxed{1} & -2 \end{array}\right] \xrightarrow{(-1)R_2\to R_2} \left[\begin{array}{ccc|c} \boxed{1} & -4 & -2 & 1 \\ 0 & \boxed{1} & -5 & 10 \\ 0 & 0 & \boxed{1} & -2 \end{array}\right]$$

$$\xrightarrow[2R_3+R_1\to R_1]{5R_3+R_2\to R_2} \left[\begin{array}{ccc|c} \boxed{1} & -4 & 0 & -3 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & -2 \end{array}\right] \xrightarrow{4R_2+R_1\to R_1} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & -3 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & -2 \end{array}\right] = \left[\, I \mid [\mathbf{x}]_{\mathcal{B}} \,\right]$$

$$\therefore \quad \boxed{[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -3 \\ 0 \\ -2 \end{bmatrix}} \quad \text{OR} \quad \boxed{[\mathbf{x}]_{\mathcal{B}} = (-3, 0, -2)^T}$$

BE CAREFUL: $[\mathbf{x}]_{\mathcal{B}}$ cannot be written as $-3\widehat{\mathbf{i}} - 2\widehat{\mathbf{k}}$ since basis $\mathcal{B}$ is **non-standard!**

EX 4.7.3: Let bases $\mathcal{B} = \left\{ \begin{bmatrix} 5 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -6 \\ 1 \\ 14 \end{bmatrix} \right\}$ and $\mathcal{B}' = \left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ -9 \\ 12 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} \right\}$.

Find $[\mathbf{x}]_{\mathcal{B}'}$ if $[\mathbf{x}]_{\mathcal{B}} = (1, -2, 1)^T$.

$$[\mathbf{x}]_{\mathcal{B}} = (1)\begin{bmatrix} 5 \\ -4 \\ 2 \end{bmatrix} + (-2)\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} + (1)\begin{bmatrix} -6 \\ 1 \\ 14 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \\ 10 \end{bmatrix} \Longrightarrow \boxed{[\mathbf{x}]_{\mathcal{E}} = \begin{bmatrix} -3 \\ -1 \\ 10 \end{bmatrix}}$$

$$\left[\, \mathcal{B}' \mid [\mathbf{x}]_{\mathcal{E}} \,\right] = \left[\begin{array}{ccc|c} \boxed{1} & -4 & -2 & -3 \\ 2 & -9 & 1 & -1 \\ -3 & 12 & 7 & 10 \end{array}\right] \xrightarrow[3R_1+R_3\to R_3]{(-2)R_1+R_2\to R_2} \left[\begin{array}{ccc|c} \boxed{1} & -4 & -2 & -3 \\ 0 & -1 & 5 & 5 \\ 0 & 0 & \boxed{1} & 1 \end{array}\right] \xrightarrow{(-1)R_2\to R_2} \left[\begin{array}{ccc|c} \boxed{1} & -4 & -2 & -3 \\ 0 & \boxed{1} & -5 & -5 \\ 0 & 0 & \boxed{1} & 1 \end{array}\right]$$

$$\xrightarrow[2R_3+R_1\to R_1]{5R_3+R_2\to R_2} \left[\begin{array}{ccc|c} \boxed{1} & -4 & 0 & -1 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 1 \end{array}\right] \xrightarrow{4R_2+R_1\to R_1} \left[\begin{array}{ccc|c} \boxed{1} & 0 & 0 & -1 \\ 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 1 \end{array}\right] = \left[\, I \mid [\mathbf{x}]_{\mathcal{B}'} \,\right]$$

$$\therefore \quad \boxed{[\mathbf{x}]_{\mathcal{B}'} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}} \quad \text{OR} \quad \boxed{[\mathbf{x}]_{\mathcal{B}'} = (-1, 0, 1)^T}$$

BE CAREFUL: $[\mathbf{x}]_{\mathcal{B}'}$ cannot be written as $-\widehat{\mathbf{i}} + \widehat{\mathbf{k}}$ since basis $\mathcal{B}'$ is **non-standard!**

EX 4.7.4: Let bases $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} \right\}$ and $\mathcal{B}' = \left\{ \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(a) Find the transition matrix $\underset{\mathcal{B}' \leftarrow \mathcal{B}}{P}$ from basis $\mathcal{B}$ to basis $\mathcal{B}'$.

$$[\mathcal{B}' \mid \mathcal{B}] = \left[\begin{array}{ccc|ccc} 0 & -2 & 1 & 1 & -1 & 2 \\ 2 & 1 & 1 & 2 & 2 & 4 \\ 1 & 0 & 1 & 4 & 0 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_3} \left[\begin{array}{ccc|ccc} \boxed{1} & 0 & 1 & 4 & 0 & 0 \\ 2 & 1 & 1 & 2 & 2 & 4 \\ 0 & -2 & 1 & 1 & -1 & 2 \end{array}\right] \xrightarrow{(-2)R_1 + R_2 \to R_2} \left[\begin{array}{ccc|ccc} \boxed{1} & 0 & 1 & 4 & 0 & 0 \\ 0 & \boxed{1} & -1 & -6 & 2 & 4 \\ 0 & -2 & 1 & 1 & -1 & 2 \end{array}\right]$$

$$\xrightarrow{2R_2 + R_3 \to R_3} \left[\begin{array}{ccc|ccc} \boxed{1} & 0 & 1 & 4 & 0 & 0 \\ 0 & \boxed{1} & -1 & -6 & 2 & 4 \\ 0 & 0 & -1 & -11 & 3 & 10 \end{array}\right] \xrightarrow{(-1)R_3 \to R_3} \left[\begin{array}{ccc|ccc} \boxed{1} & 0 & 1 & 4 & 0 & 0 \\ 0 & \boxed{1} & -1 & -6 & 2 & 4 \\ 0 & 0 & \boxed{1} & 11 & -3 & -10 \end{array}\right]$$

$$\xrightarrow[(-1)R_3 + R_1 \to R_1]{R_3 + R_2 \to R_2} \left[\begin{array}{ccc|ccc} \boxed{1} & 0 & 0 & -7 & 3 & 10 \\ 0 & \boxed{1} & 0 & 5 & -1 & -6 \\ 0 & 0 & \boxed{1} & 11 & -3 & -10 \end{array}\right] = \left[I \;\middle|\; \underset{\mathcal{B}' \leftarrow \mathcal{B}}{P} \right] \qquad \therefore \boxed{\underset{\mathcal{B}' \leftarrow \mathcal{B}}{P} = \begin{bmatrix} -7 & 3 & 10 \\ 5 & -1 & -6 \\ 11 & -3 & -10 \end{bmatrix}}$$

(b) Find $[\mathbf{x}]_{\mathcal{B}'}$ if $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$.

$$[\mathbf{x}]_{\mathcal{B}'} = \underset{\mathcal{B}' \leftarrow \mathcal{B}}{P} [\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} -7 & 3 & 10 \\ 5 & -1 & -6 \\ 11 & -3 & -10 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} -5 \\ 7 \\ 13 \end{bmatrix}}$$

(c) Find the transition matrix $\underset{\mathcal{B} \leftarrow \mathcal{B}'}{P}$ from basis $\mathcal{B}'$ to basis $\mathcal{B}$.

$$[\mathcal{B} \mid \mathcal{B}'] = \left[\begin{array}{ccc|ccc} \boxed{1} & -1 & 2 & 0 & -2 & 1 \\ 2 & 2 & 4 & 2 & 1 & 1 \\ 4 & 0 & 0 & 1 & 0 & 1 \end{array}\right] \xrightarrow[(-4)R_1 + R_3 \to R_3]{(-2)R_1 + R_2 \to R_2} \left[\begin{array}{ccc|ccc} \boxed{1} & -1 & 2 & 0 & -2 & 1 \\ 0 & 4 & 0 & 2 & 5 & -1 \\ 0 & 4 & -8 & 1 & 8 & -3 \end{array}\right]$$

$$\xrightarrow{(-1)R_2 + R_3 \to R_3} \left[\begin{array}{ccc|ccc} \boxed{1} & -1 & 2 & 0 & -2 & 1 \\ 0 & 4 & 0 & 2 & 5 & -1 \\ 0 & 0 & -8 & -1 & 3 & -2 \end{array}\right] \xrightarrow[(-\frac{1}{8})R_3 \to R_3]{(\frac{1}{4})R_2 \to R_2} \left[\begin{array}{ccc|ccc} \boxed{1} & -1 & 2 & 0 & -2 & 1 \\ 0 & \boxed{1} & 0 & 1/2 & 5/4 & -1/4 \\ 0 & 0 & \boxed{1} & 1/8 & -3/8 & 1/4 \end{array}\right]$$

$$\xrightarrow{(-2)R_3 + R_1 \to R_1} \left[\begin{array}{ccc|ccc} \boxed{1} & -1 & 0 & -1/4 & -5/4 & 1/2 \\ 0 & \boxed{1} & 0 & 1/2 & 5/4 & -1/4 \\ 0 & 0 & \boxed{1} & 1/8 & -3/8 & 1/4 \end{array}\right] \xrightarrow{R_2 + R_1 \to R_1} \left[\begin{array}{ccc|ccc} \boxed{1} & 0 & 0 & 1/4 & 0 & 1/4 \\ 0 & \boxed{1} & 0 & 1/2 & 5/4 & -1/4 \\ 0 & 0 & \boxed{1} & 1/8 & -3/8 & 1/4 \end{array}\right] = \left[I \;\middle|\; \underset{\mathcal{B} \leftarrow \mathcal{B}'}{P} \right]$$

$$\therefore \boxed{\underset{\mathcal{B} \leftarrow \mathcal{B}'}{P} = \begin{bmatrix} 1/4 & 0 & 1/4 \\ 1/2 & 5/4 & -1/4 \\ 1/8 & -3/8 & 1/4 \end{bmatrix}}$$

(d) Find $[\mathbf{x}]_{\mathcal{B}}$ if $[\mathbf{x}]_{\mathcal{B}'} = (2, 3, 0)^T$.

$$[\mathbf{x}]_{\mathcal{B}} = \underset{\mathcal{B} \leftarrow \mathcal{B}'}{P} [\mathbf{x}]_{\mathcal{B}'} = \begin{bmatrix} 1/4 & 0 & 1/4 \\ 1/2 & 5/4 & -1/4 \\ 1/8 & -3/8 & 1/4 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} 1/2 \\ 19/4 \\ -7/8 \end{bmatrix}}$$