CHANGE OF BASIS: COORD. VECTOR, TRANSITION MATRIX [LARSON 4.7]

- **ASSUMPTIONS MADE THROUGHOUT THIS SECTION:**

  Let V be a **finite-dimensional** vector space.

  Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be an ordered basis for V.

  Let $\mathcal{B}' = \{\mathbf{v}'_1, \mathbf{v}'_2, \ldots, \mathbf{v}'_n\}$ be another ordered basis for V.

  Let $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be the ordered **standard basis** for V.

- **COORDINATE VECTOR RELATIVE TO A BASIS:**

  Let vector $\mathbf{x} \in V$ s.t. $\mathbf{x} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$

  Then the **coordinate vector of x relative to basis** $\mathcal{B}$ is $[\mathbf{x}]_{\mathcal{B}} = (c_1, c_2, \ldots, c_n)^T$

  where $c_1, c_2, \ldots, c_n$ are the **coordinates of x relative to basis** $\mathcal{B}$.

  $[\mathbf{x}]_{\mathcal{B}}$ is also known as the $\mathcal{B}$**-coordinate vector of x**.

  $c_1, c_2, \ldots, c_n$ are also known as the $\mathcal{B}$**-coordinates of x**.

  The order of the vectors in the basis is critical, hence the term **ordered basis**.

- **CONVERTING $[\mathbf{x}]_{\mathcal{B}} \to [\mathbf{x}]_{\mathcal{E}}$ (PROCEDURE):**

  Let $[\mathbf{x}]_{\mathcal{B}} = (c_1, c_2, \ldots, c_n)^T$.

  Then $[\mathbf{x}]_{\mathcal{E}} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n \quad \leftarrow$ (Simplify linear combination)

- **CONVERTING $[\mathbf{x}]_{\mathcal{E}} \to [\mathbf{x}]_{\mathcal{B}}$ (PROCEDURE):**

  GIVEN: Vector **x** in standard basis coordinates: $[\mathbf{x}]_{\mathcal{E}}$

  TASK: Write vector **x** in non-std basis $\mathcal{B}$-coordinates: $[\mathbf{x}]_{\mathcal{B}}$

  (1) $[\mathcal{B} \mid [\mathbf{x}]_{\mathcal{E}}] \xrightarrow{Gauss-Jordan} [\, I \mid [\mathbf{x}]_{\mathcal{B}}]$

- **CONVERTING $[\mathbf{x}]_{\mathcal{B}} \to [\mathbf{x}]_{\mathcal{B}'}$ (PROCEDURE):**

  GIVEN: Vector **x** in non-std basis $\mathcal{B}$-coordinates: $[\mathbf{x}]_{\mathcal{B}}$

  TASK: Write vector **x** in non-std basis $\mathcal{B}'$-coordinates: $[\mathbf{x}]_{\mathcal{B}'}$

  (1) Convert $[\mathbf{x}]_{\mathcal{B}} \to [\mathbf{x}]_{\mathcal{E}}$

  (2) $[\mathcal{B}' \mid [\mathbf{x}]_{\mathcal{E}}] \xrightarrow{Gauss-Jordan} [\, I \mid [\mathbf{x}]_{\mathcal{B}'}]$

- **TRANSITION MATRIX (DEFINITION):**

  The **transition matrix** $\underset{\mathcal{B}'\leftarrow\mathcal{B}}{P}$ **from** $\mathcal{B}$ to $\mathcal{B}'$ satisfies $\underset{\mathcal{B}'\leftarrow\mathcal{B}}{P}[\mathbf{x}]_{\mathcal{B}} = [\mathbf{x}]_{\mathcal{B}'}$

  The **transition matrix** $\underset{\mathcal{B}\leftarrow\mathcal{B}'}{P}$ **from** $\mathcal{B}'$ to $\mathcal{B}$ satisfies $\underset{\mathcal{B}\leftarrow\mathcal{B}'}{P}[\mathbf{x}]_{\mathcal{B}'} = [\mathbf{x}]_{\mathcal{B}}$

- **INVERSE OF A TRANSITION MATRIX:**

  $$\left(\underset{\mathcal{B}\leftarrow\mathcal{B}'}{P}\right)^{-1} = \underset{\mathcal{B}'\leftarrow\mathcal{B}}{P} \quad \text{and} \quad \left(\underset{\mathcal{B}'\leftarrow\mathcal{B}}{P}\right)^{-1} = \underset{\mathcal{B}\leftarrow\mathcal{B}'}{P}$$

- **FINDING THE TRANSITION MATRIX (PROCEDURE):**

  TASK: Find the transition matrix $\underset{\mathcal{B}'\leftarrow\mathcal{B}}{P}$ from $\mathcal{B}$ to $\mathcal{B}'$.

  (1) $[\mathcal{B}' \mid \mathcal{B}] \xrightarrow{Gauss-Jordan} \left[\, I \;\middle|\; \underset{\mathcal{B}'\leftarrow\mathcal{B}}{P} \right]$

EX 4.7.1: Let basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ -9 \\ 12 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} \right\} \equiv \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Find $[\mathbf{x}]_{\mathcal{E}}$ if $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}$.

EX 4.7.2: Let basis $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ -9 \\ 12 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} \right\} \equiv \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Find $[\mathbf{x}]_{\mathcal{B}}$ if $[\mathbf{x}]_{\mathcal{E}} = \begin{bmatrix} 1 \\ -8 \\ -5 \end{bmatrix}$.

EX 4.7.3: Let bases $\mathcal{B} = \left\{ \begin{bmatrix} 5 \\ -4 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -6 \\ 1 \\ 14 \end{bmatrix} \right\}$ and $\mathcal{B}' = \left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ -9 \\ 12 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 7 \end{bmatrix} \right\}$.

Find $[\mathbf{x}]_{\mathcal{B}'}$ if $[\mathbf{x}]_{\mathcal{B}} = (1, -2, 1)^T$.

EX 4.7.4: Let bases $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} \right\}$ and $\mathcal{B}' = \left\{ \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

(a) Find the transition matrix $\underset{\mathcal{B}' \leftarrow \mathcal{B}}{P}$ from basis $\mathcal{B}$ to basis $\mathcal{B}'$.

(b) Find $[\mathbf{x}]_{\mathcal{B}'}$ if $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}$.

(c) Find the transition matrix $\underset{\mathcal{B} \leftarrow \mathcal{B}'}{P}$ from basis $\mathcal{B}'$ to basis $\mathcal{B}$.

(d) Find $[\mathbf{x}]_{\mathcal{B}}$ if $[\mathbf{x}]_{\mathcal{B}'} = (2, 3, 0)^T$.