VECTORS: NORMS, DOT PRODUCTS, PROJECTIONS [LARSON 5.1]

- **NORMS OF VECTORS:** The **norm** of a vector is simply its "length" or "magnitude"
  - The **norm** of vector $\mathbf{v} = (v_1, v_2)^T \in \mathbb{R}^2$ is: $||\mathbf{v}|| := \sqrt{v_1^2 + v_2^2}$
  - The **norm** of vector $\mathbf{v} = (v_1, v_2, v_3)^T \in \mathbb{R}^3$ is: $||\mathbf{v}|| := \sqrt{v_1^2 + v_2^2 + v_3^2}$
  - The **norm** of vector $\mathbf{v} = (v_1, v_2, \cdots, v_n)^T \in \mathbb{R}^n$ is: $||\mathbf{v}|| := \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$
- **UNIT VECTORS (AKA DIRECTION VECTORS):** A **unit vector** $\widehat{\mathbf{v}}$ is a vector with **norm one**.
  - A **unit vector** (AKA **direction vector**) for vector $\mathbf{v}$ is: $\widehat{\mathbf{v}} := \dfrac{\mathbf{v}}{||\mathbf{v}||}$

- **DOT PRODUCTS OF VECTORS:**
  - The **dot product** of $\mathbf{v} = (v_1, v_2)^T$ and $\mathbf{w} = (w_1, w_2)^T$ is: $\mathbf{v} \cdot \mathbf{w} := \mathbf{v}^T\mathbf{w} = v_1w_1 + v_2w_2$
  - The **dot product** of $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$ and $\mathbf{w} = (w_1, w_2, w_3)^T$ is: $\mathbf{v} \cdot \mathbf{w} := \mathbf{v}^T\mathbf{w} = v_1w_1 + v_2w_2 + v_3w_3$
  - The **dot product** of $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ and $\mathbf{w} = (w_1, w_2, \cdots, w_n)^T$ is: $\mathbf{v} \cdot \mathbf{w} := \mathbf{v}^T\mathbf{w} = v_1w_1 + v_2w_2 + \cdots + v_nw_n$
- **PROPERTIES OF DOT PRODUCTS:** Let vectors $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and scalar $\alpha \in \mathbb{R}$. Then:

(DP1)	$\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$	Commutativity of Dot Product
(DP2)	$\alpha(\mathbf{v} \cdot \mathbf{w}) = (\alpha\mathbf{v}) \cdot \mathbf{w} = \mathbf{v} \cdot (\alpha\mathbf{w})$	Associativity of Dot Product
(DP3)	$\vec{\mathbf{0}} \cdot \mathbf{v} = \mathbf{v} \cdot \vec{\mathbf{0}} = 0$	Dot Product with $\vec{\mathbf{0}}$ is Zero Scalar
(DP4)	$\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$	Distributivity of Dot Product over VA
(DP5)	$\mathbf{v} \cdot \mathbf{v} = \lvert\lvert\mathbf{v}\rvert\rvert^2$	Dot Product-Norm Relationship

- **COORDINATE-FREE DEFINITION OF DOT PRODUCT:** Let vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. Then:

$$\mathbf{v} \cdot \mathbf{w} = ||\mathbf{v}||\,||\mathbf{w}|| \cos\theta \qquad \text{where } \theta \in [0, \pi]$$

- **ORTHOGONALITY OF VECTORS:**

Vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ are **orthogonal** $\iff \mathbf{v} \perp \mathbf{w} \iff \mathbf{v} \cdot \mathbf{w} = 0 \iff \mathbf{v}^T\mathbf{w} = 0$

- **(ORTHOGONAL) PROJECTION ONTO A VECTOR:** Let vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ such that $\mathbf{w} \neq \vec{\mathbf{0}}$. Then:

The **(orthogonal) projection of v onto w** is: $\text{proj}_{\mathbf{w}}\mathbf{v} := \left(\dfrac{\mathbf{v} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}}\right)\mathbf{w} = \left(\dfrac{\mathbf{v}^T\mathbf{w}}{\mathbf{w}^T\mathbf{w}}\right)\mathbf{w}$

EX 5.1.1: Let vectors $\mathbf{v} = (1, 2)^T$ and $\mathbf{w} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$.

(a) Compute dot product $\mathbf{v} \cdot \mathbf{w}$.

(b) Are $\mathbf{v}$ & $\mathbf{w}$ orthogonal?

(c) Compute $||\mathbf{v}||$.

(d) Find unit vector $\widehat{\mathbf{v}}$.

(e) Compute orthogonal projections $\text{proj}_{\mathbf{w}}\mathbf{v}$ and $\text{proj}_{\mathbf{v}}\mathbf{w}$.

EX 5.1.2: Let vectors $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{w} = (4, -5, -6)^T$.

(a) Compute dot product $\mathbf{v}^T\mathbf{w}$.

(b) Are $\mathbf{v}$ & $\mathbf{w}$ orthogonal?

(c) Compute $||\mathbf{w}||$.

(d) Find unit vector $\widehat{\mathbf{w}}$.

(e) Compute orthogonal projections $\text{proj}_{\mathbf{w}}\mathbf{v}$ and $\text{proj}_{\mathbf{v}}\mathbf{w}$.

EX 5.1.3: Let vectors $\mathbf{u} = (1, 2, 3, 4)^T$ and $\mathbf{v} = (-3, 3, 0, -1)^T$.

(a) Compute dot product $\mathbf{v}^T\mathbf{u}$.

(b) Are $\mathbf{u}$ & $\mathbf{v}$ orthogonal?

(c) Compute $||\mathbf{v}||$.

(d) Find unit vector $\widehat{\mathbf{v}}$.

(e) Compute orthogonal projections $\text{proj}_{\mathbf{u}}\mathbf{v}$ and $\text{proj}_{\mathbf{v}}\mathbf{u}$.